BemandaSTEM Academy

Mathematics for every Ethiopian student

GRADE 11

ALGEBRA 2

Companion Workbook

36-Week Course

10 Units · 180+ Worked Examples · 540+ Practice Problems · 288 Quiz Questions

Companion to the BemandaSTEM Grade 11 Algebra 2 offline learning platform

How to Use This Workbook

This workbook accompanies the BemandaSTEM Grade 11 Algebra 2 platform. Every week's lesson in the app has a matching chapter in this workbook, with the same examples, more practice problems, and a quiz identical to the one in the app — so you can study with or without a computer.

STRUCTURE OF EACH WEEK

Each weekly chapter follows the same six-part structure:

- Lesson Overview: the core idea and what you will learn.
- Key Concepts: definitions, formulas, and the "did you know" cultural connection.
- Worked Examples: five fully solved problems with step-by-step reasoning.
- Practice Problems: 15 problems with space to show your work.
- Self-Check Quiz: eight multiple-choice questions to test your understanding.
- Self-Assessment: a checklist to track your mastery.

TIPS FOR SUCCESS

Always read the lesson overview first, then study the worked examples carefully before attempting practice problems. Do not skip the steps — write them out, even when you think you know the answer.

When you finish a week, take the self-check quiz without looking at your notes. If you score below 80 percent, revisit the lesson and try again.

At the end of each unit, an answer key shows the correct answers and brief explanations for every quiz question.

SYMBOLS USED IN THIS WORKBOOK

★ Key formula or concept to memorize

▶ Worked example with full solution

✎ Practice problem — show your work

? Self-check quiz question

✓ Self-assessment checklist item

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Unit 8: Systems of Equations & Inequalities

Unit 9: Sequences & Series

Unit 10: Probability & Final Review

UNIT 1

Review of Algebra 1 Concepts

WEEKS COVERED

Week 1: Real Numbers and Expressions
Week 2: Linear Equations and Inequalities
Week 3: Review and Real-World Applications

Unit 1 · Week 1

Real Numbers and Expressions

Discover how every number you encounter — from counting goats in Lalibela to measuring the curvature of the Blue Nile — fits into the family of real numbers.

LESSON OVERVIEW

Welcome to Algebra 2. Today we begin with the foundation of all algebra: real numbers.

Imagine an Ethiopian farmer counting injera baskets at a market in Mercato. One, two, three baskets. These are natural numbers — the counting numbers.

Now include zero. Zero baskets means none. Together with the natural numbers, zero gives us the whole numbers.

But what about debt? If the farmer owes three birr, we write negative three. Adding negatives to whole numbers gives us the integers.

When we cut one injera into four equal pieces, each piece is one quarter, or 0.25. Numbers like this — fractions and ratios — are called rational numbers.

But some numbers cannot be written as fractions. The number pi, the ratio of a circle to its diameter, goes on forever without repeating. So does the square root of two. These are called irrational numbers.

All of these together — naturals, wholes, integers, rationals, and irrationals — form the family of real numbers.

An algebraic expression combines numbers, variables, and operations. For example, three x plus five. The letter x is a variable. The number three is the coefficient. The number five is the constant.

Today, you will learn to classify numbers, evaluate expressions using the order of operations, and translate words into algebra.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

Natural ⊂ Whole ⊂ Integers ⊂ Rationals ⊂ Reals

💡 **DID YOU KNOW?**

The ancient Ethiopian numbering system, the Geez numerals, has been used for over 1,500 years. The Aksumite Empire developed sophisticated arithmetic long before algebra reached Europe.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1 Classify -8

Step 1	It is negative and whole-valued
Step 2	It belongs to integers
Step 3	It is also rational and real

ANSWER Integer, Rational, Real

EXAMPLE 2 Classify 0.75

Step 1	Write as a fraction: 0.75 = 3/4
Step 2	Since it can be written as a fraction, it is rational

ANSWER Rational

EXAMPLE 3 Evaluate 6 + 2(4)

Step 1	Multiply first: $2 \times 4 = 8$
Step 2	Then add: $6 + 8 = 14$

ANSWER 14

EXAMPLE 4 Simplify 3(x + 2)

Step 1	Use the distributive property
Step 2	Multiply 3 by each term inside

ANSWER **$3x + 6$**

EXAMPLE 5 **Translate "Twice a number decreased by 4"**

Step 1	Let the number = x
Step 2	Twice a number = 2x
Step 3	Decreased by 4 means subtract 4

ANSWER **$2x - 4$**

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 Classify each number: 7, −3, 0.5, √2, π, 22/7

2 Evaluate: $6 + 4 \times 3 - 2$

3 Evaluate: $20 - (4 + 6) \div 2$

4 Simplify: $5(x + 7)$

5 Simplify: $-3(2x - 4)$

6

Translate "nine more than twice a number" into an algebraic expression.

7

Translate "the difference between a number and 11" into an algebraic expression.

8

Which set contains all integers AND fractions? (a) Natural (b) Whole (c) Rational (d) Irrational

9

Evaluate: $3^2 + 4^2$

10

Simplify: $2(3x - 1) + 5x$

11

Is $\sqrt{9}$ rational or irrational? Explain.

12

Evaluate: $8 - 2 \times 3 + 1$

13

Translate "five less than three times a number" into an expression.

14

Evaluate: $(5 + 3)^2 \div 4$

15

Simplify: $4x + 7 - 2x - 3$

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. Which set does the number -5 belong to?
- **A.** Natural numbers only
- **B.** Whole numbers
- **C.** Integers, rational, real
- **D.** Irrational

2. Classify the number $\sqrt{7}$.
- **A.** Rational
- **B.** Irrational
- **C.** Whole
- **D.** Integer

3. Evaluate: $4 + 3 \times 2$
- **A.** 14
- **B.** 10
- **C.** 7
- **D.** 6

4. Simplify: $5(x + 3)$
- **A.** $5x + 3$
- **B.** $x + 15$
- **C.** $5x + 15$
- **D.** $5x + 8$

5. Which property says $a + b = b + a$?
- **A.** Associative
- **B.** Distributive
- **C.** Commutative
- **D.** Identity

6. Translate: "Seven more than a number"
- **A.** $7 - x$
- **B.** $x - 7$
- **C.** $7x$
- **D.** $x + 7$

7. In the expression $9y$, what is the coefficient?
- **A.** y
- **B.** 9
- **C.** $9y$

- **D.** none

8. Which number is irrational?

- **A.** 1/3
- **B.** -4
- **C.** π
- **D.** 0.5

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can classify numbers as natural, whole, integer, rational, irrational, or real.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can evaluate expressions using the order of operations (PEMDAS).
Confident ☐ Mostly ☐ Need Review ☐

✓ I can use the distributive property to simplify expressions.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can translate word phrases into algebraic expressions.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can apply real-number concepts to real-world problems.
Confident ☐ Mostly ☐ Need Review ☐

Unit 1 · Week 2

Linear Equations and Inequalities

Learn to solve equations like a true scholar — balancing both sides like the scales of justice at Fasilides Castle.

LESSON OVERVIEW

Welcome back. This week, we balance equations like the merchants of Aksum balanced their scales.

An equation is a statement that two expressions are equal. For example, x plus five equals twelve.

A linear equation has its variable raised only to the first power. There are no x-squared or x-cubed terms.

To solve an equation means to find the value of x that makes the statement true.

Think of an equation like a balance scale. Whatever you do to one side, you must do to the other side. This keeps the balance.

The trick is to use inverse operations. Addition undoes subtraction. Multiplication undoes division. We use them to isolate x on one side.

An inequality is similar, but uses less-than, greater-than, or equal-to symbols instead of an equals sign.

One very important rule: when you multiply or divide both sides of an inequality by a negative number, you must reverse the inequality sign.

Today you will solve one-step, two-step, and multi-step equations, and graph inequality solutions on a number line.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

`To solve: do the same operation to BOTH sides`

💡 **DID YOU KNOW?**

The word "algebra" comes from the Arabic word "al-jabr", meaning "the reunion of broken parts" — describing the process of moving terms to balance an equation.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1	**Solve $x + 7 = 15$**
Step 1	Subtract 7 from both sides
Step 2	$x = 15 - 7$
ANSWER	**$x = 8$**

EXAMPLE 2	**Solve $3x + 4 = 19$**
Step 1	Subtract 4: $3x = 15$
Step 2	Divide by 3: $x = 5$
ANSWER	**$x = 5$**

EXAMPLE 3	**Solve $5x - 2 = 2x + 7$**
Step 1	Subtract 2x: $3x - 2 = 7$
Step 2	Add 2: $3x = 9$
Step 3	Divide by 3: $x = 3$
ANSWER	**$x = 3$**

EXAMPLE 4	**Solve $x - 4 \geq 6$**
Step 1	Add 4 to both sides
Step 2	$x \geq 10$

ANSWER $x \geq 10$

EXAMPLE 5 **Solve $-3x < 12$**

Step 1	Divide by −3 (FLIP the sign!)
Step 2	$x > -4$

ANSWER $x > -4$

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 Solve: $x + 7 = 15$

2 Solve: $3x = 21$

3 Solve: $2x + 5 = 13$

4 Solve: $5x - 8 = 17$

5 Solve: $4x + 3 = 2x + 11$

6

Solve: $3(x - 2) = 12$

7

Solve: $7x - 5 = 4x + 10$

8

Solve: $2(x + 4) = 18$

9 Solve: $6x = 4x + 14$

10 Solve: $x/3 + 2 = 7$

11 Solve: $5(2x - 1) = 25$

12 Solve: $-3x + 8 = 20$

13 Solve: $4x - 6 = 10$

14 Solve: $2x + 9 = 5x - 6$

15

Solve: $8 = 3x - 7$

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. Solve: $x + 6 = 14$
- **A.** $x = 8$
- **B.** $x = 20$
- **C.** $x = 6$
- **D.** $x = -8$

2. Solve: $4x = 32$
- **A.** $x = 36$
- **B.** $x = 8$
- **C.** $x = 28$
- **D.** $x = 128$

3. Solve: $2x + 5 = 15$
- **A.** $x = 10$
- **B.** $x = 5$
- **C.** $x = 20$
- **D.** $x = 7.5$

4. Solve: $7x - 3 = 4x + 9$
- **A.** $x = 4$
- **B.** $x = 3$
- **C.** $x = 2$
- **D.** $x = 6$

5. Solve: $x - 5 < 3$
- **A.** $x < -2$
- **B.** $x < 8$
- **C.** $x > 8$
- **D.** $x > -2$

6. Solve: $-2x > 10$
- **A.** $x > -5$
- **B.** $x < -5$
- **C.** $x > 5$
- **D.** $x < 5$

7. Which symbol means "greater than or equal to"?
- **A.** $<$
- **B.** $>$
- **C.** $\leq$

- **D.** ≥

8. On a number line graph of $x \geq 4$, the circle at 4 is:

- **A.** Open
- **B.** Closed
- **C.** Both
- **D.** Neither

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can solve one-step and two-step linear equations.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can solve multi-step equations with variables on both sides.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can solve equations involving the distributive property.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can solve and graph linear inequalities on a number line.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can apply linear equations and inequalities to real-world contexts.
Confident ☐ Mostly ☐ Need Review ☐

Unit 1 · Week 3

Review and Real-World Applications

Bring your skills together in true Ethiopian style — solving real-life problems from the marketplaces of Addis Ababa to the highlands of Tigray.

LESSON OVERVIEW

Welcome to Week 3. This is a review week, where we apply everything we have learned to real-life situations.

Algebra is not just for the classroom. It is the secret language behind every market price, every road trip, and every savings plan.

Imagine you are saving birr each month to buy a bicycle. If you save fifty birr each week, how many weeks until you reach 800 birr? That is an algebra problem.

Imagine you are travelling from Addis Ababa to Bahir Dar, 565 kilometres away. If your bus travels at 80 kilometres per hour, how long will the trip take? That is an algebra problem too.

This week we will review classifying numbers, evaluating expressions, solving equations and inequalities, and translating words into algebra.

Real-world problems require careful reading. Identify what you know, what you want to find, choose an equation, solve step by step, and check your answer against the situation.

By the end of this week, you will be ready for the Unit 1 final assessment.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

Read · Identify · Set up · Solve · Check

💡 **DID YOU KNOW?**

Traditional Ethiopian coffee ceremonies use precise measurements — the equivalent of algebra in daily life. Each cup is measured by the host using ratios passed down through generations.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1 Classify -12

Step 1	Negative and whole-valued
Step 2	It is an integer
Step 3	Also rational and real

ANSWER Integer, Rational, Real

EXAMPLE 2 Evaluate $2x + 5$ when $x = 3$

Step 1	Substitute: $2(3) + 5$
Step 2	Multiply: $6 + 5$
Step 3	Add: 11

ANSWER 11

EXAMPLE 3 Solve $5x - 7 = 18$

Step 1	Add 7: $5x = 25$
Step 2	Divide by 5: $x = 5$

ANSWER $x = 5$

EXAMPLE 4 Solve $3x + 2 > 14$

Step 1	Subtract 2: $3x > 12$

Step 2	Divide by 3: $x > 4$
ANSWER	$x > 4$

EXAMPLE 5 A gym charges 15 birr monthly plus 8 birr per class. Write an expression for x classes.

Step 1	Monthly fee: 15
Step 2	Class cost: $8x$
Step 3	Total: $15 + 8x$
ANSWER	$15 + 8x$

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1

Solve: $5x - 12 = 23$

2

Simplify: $4(x - 3) + 2x$

3

Classify the number 0.333... (repeating).

4

Solve: $3(2x + 1) = 21$

5

Evaluate: $10 - 2^3 + 4$

6

Translate "twice the sum of a number and 5".

7

Solve: $x/4 - 1 = 5$

8

Simplify: $7y - 3(y + 2)$

9

Is $\pi/2$ rational or irrational?

10

Solve: $6x + 4 = 2x + 24$

11

Evaluate: $(12 - 4) \times 2 + 1$

12

Simplify: $-2(3x - 5) + 4x$

13

Solve: $9 - 2x = 1$

14

Classify: $-7/3$

15

Solve: $4(x + 3) = 28$

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. Classify the number 0.

- **A.** Natural only
- **B.** Whole, integer, rational, real
- **C.** Irrational
- **D.** Negative

2. Evaluate 4x + 2 when x = 5.

- **A.** 22
- **B.** 20
- **C.** 25
- **D.** 12

3. Solve: 6x − 3 = 21

- **A.** x = 4
- **B.** x = 3
- **C.** x = 5
- **D.** x = 6

4. Solve: x − 5 ≤ 8

- **A.** x ≤ 13
- **B.** x ≤ 3
- **C.** x ≥ 13
- **D.** x ≥ 3

5. A phone plan costs 25 birr plus 10 birr per extra GB used. Expression for x extra GB?

- **A.** 25x + 10
- **B.** 35x
- **C.** 25 + 10x
- **D.** 10 + 25x

6. Solve: 4x + 7 = 3x + 15

- **A.** x = 22
- **B.** x = 8
- **C.** x = 7
- **D.** x = 11

7. Which number is rational but NOT an integer?

- **A.** -3
- **B.** 0

- **C.** $\sqrt{2}$
- **D.** 0.75

8. You have 40 birr. You spend x and need at least 10 left. The inequality is:

- **A.** $40 - x \geq 10$
- **B.** $40 + x \geq 10$
- **C.** $x - 40 \geq 10$
- **D.** $x \geq 30$

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can apply algebra to real-world word problems.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can set up and solve linear equations from descriptive scenarios.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can interpret solutions in real-world contexts (units, reasonableness).
Confident ☐ Mostly ☐ Need Review ☐

✓ I can use inequalities to express real-world constraints.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can confidently combine all Unit 1 skills.
Confident ☐ Mostly ☐ Need Review ☐

Unit 1 Answer Key

Compare your answers below. If you missed a question, review the corresponding lesson section.

Week 1: Real Numbers and Expressions

Q#	Answer	Explanation
1	**C**	-5 is negative, so it is not a natural or whole number. It is an integer, which is also rational and real.
2	**B**	$\sqrt{7}$ cannot be written as a fraction, so it is irrational.
3	**B**	Order of operations: multiply first ($3\times2=6$), then add ($4+6=10$).
4	**C**	Distribute: $5\cdot x + 5\cdot 3 = 5x + 15$.
5	**C**	The commutative property says order does not matter for addition or multiplication.
6	**D**	"More than" means addition. Seven more than x is $x + 7$.
7	**B**	The coefficient is the numerical factor in front of the variable.
8	**C**	π has a decimal that never ends or repeats, so it is irrational.

Week 2: Linear Equations and Inequalities

Q#	Answer	Explanation
1	**A**	Subtract 6 from both sides: $x = 14 - 6 = 8$.
2	**B**	Divide both sides by 4: $x = 32 \div 4 = 8$.
3	**B**	Subtract 5: $2x = 10$. Divide by 2: $x = 5$.
4	**A**	Subtract 4x: $3x - 3 = 9$. Add 3: $3x = 12$. Divide: $x = 4$.
5	**B**	Add 5 to both sides: $x < 8$.
6	**B**	Divide by -2 and FLIP the sign: $x < -5$.
7	**D**	The symbol $\geq$ means greater than or equal to.
8	**B**	For $\leq$ or $\geq$, use a closed (filled) circle to include the endpoint.

Week 3: Review and Real-World Applications

Q#	Answer	Explanation
1	**B**	0 is a whole number, an integer, rational, and real — but not natural.
2	**A**	Substitute: $4(5) + 2 = 20 + 2 = 22$.
3	**A**	Add 3: $6x = 24$. Divide by 6: $x = 4$.
4	**A**	Add 5 to both sides: $x \le 13$.
5	**C**	Fixed 25 plus 10 times the GBs: $25 + 10x$.
6	**B**	Subtract 3x: $x + 7 = 15$. Subtract 7: $x = 8$.
7	**D**	$0.75 = 3/4$ is rational but not whole, so not an integer.
8	**A**	After spending x, you have $40 - x$. That must be at least 10: $40 - x \ge 10$.

UNIT 2

Functions

WEEKS COVERED

Week 4: Function Notation
Week 5: Domain and Range
Week 6: Types of Functions
Week 7: Piecewise Functions & Applications

Unit 2 · Week 4

Function Notation

Discover the elegant language mathematicians use to describe relationships — like the Ethiopian coffee bean: one tree gives one harvest each season.

LESSON OVERVIEW

Welcome to Unit 2. This week we enter the world of functions — the heartbeat of modern mathematics.

Imagine a coffee farmer in Yirgacheffe. Each coffee tree produces a certain amount of beans each year. If we know the number of trees, we can predict the harvest. That is a function in action.

A relation is simply a set of ordered pairs — pairs of inputs and outputs. For example, the pairs one and two, two and four, three and six form a relation.

A function is a special kind of relation: each input has exactly one output. Like the coffee tree — one tree, one harvest per season.

We write functions using special notation: f of x. This is read "f of x." It means: the function f acting on the input x.

For example, if f of x equals 2x plus 3, then f of 4 means we replace x with 4. So f of 4 equals 2 times 4 plus 3, which is 11.

The domain is the set of all possible inputs. The range is the set of all possible outputs.

Functions describe everything: how cost depends on quantity, how distance depends on time, how pay depends on hours worked.

Today you will learn to recognize functions, evaluate them, and find their domain and range.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

`f(x) means: apply function f to input x`

💡 **DID YOU KNOW?**

The notation f(x) was invented by Swiss mathematician Leonhard Euler in

1734. Before then, mathematicians had to describe functions using long sentences — Euler made math beautifully concise.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1	**Evaluate f(x) = 3x + 1 at x = 2**
Step 1	Substitute 2 for x
Step 2	f(2) = 3(2) + 1
Step 3	Multiply, then add: 6 + 1
ANSWER	**f(2) = 7**

EXAMPLE 2	**Is {(1,4),(2,5),(3,6)} a function?**
Step 1	Check inputs: 1, 2, 3
Step 2	No input is repeated
Step 3	Each input has exactly one output
ANSWER	**Yes, it is a function**

EXAMPLE 3	**Is {(2,3),(2,8),(4,1)} a function?**
Step 1	Input 2 appears twice
Step 2	It maps to both 3 and 8
Step 3	This violates the function rule
ANSWER	**Not a function**

EXAMPLE 4	**Find domain and range of {(1,5),(2,7),(4,9)}**

Step 1	Domain = all inputs = {1, 2, 4}
Step 2	Range = all outputs = {5, 7, 9}
ANSWER	**Domain {1,2,4}, Range {5,7,9}**

EXAMPLE 5	**A taxi charges 4 birr plus 3 birr per km. Find cost for 5 km.**
Step 1	Let x = km, write C(x) = 3x + 4
Step 2	Substitute 5: C(5) = 3(5) + 4
Step 3	Simplify: 15 + 4
ANSWER	**C(5) = 19 birr**

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 If $f(x) = 2x + 3$, find $f(4)$.

2 If $f(x) = 5x - 1$, find $f(0)$.

3 If $g(x) = x^2 + 2$, find $g(3)$.

4 If $h(x) = 3x - 5$, find $h(-2)$.

5 Is $\{(1,2), (2,4), (3,6)\}$ a function? Justify.

6

Is {(1,3), (1,5), (2,7)} a function? Justify.

7

Find the domain and range of {(0,1), (2,3), (4,5)}.

8

Find the domain and range of $\{(-1, 4), (0, 4), (1, 4)\}$.

9 Write a function rule: a worker earns 15 birr per hour.

10 Write a function rule: a taxi charges 8 birr base + 4 birr per km.

11 If $f(x) = 4 - x$, find $f(7)$.

12 If $g(x) = x^2 - x$, find $g(3)$.

13 For $f(x) = 2x^2$, find $f(5)$.

14 A streaming service charges 10 birr/month + 3 birr per movie. Write the function.

15

If $f(x) = 6x + 11$, find $f(-1)$.

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. If f(x) = 2x + 3, what is f(5)?

- **A.** 10
- **B.** 13
- **C.** 7
- **D.** 25

2. Is the relation {(1,2),(3,4),(5,6)} a function?

- **A.** Yes
- **B.** No
- **C.** Cannot tell
- **D.** Only sometimes

3. Is {(2,5),(2,7),(3,9)} a function?

- **A.** Yes
- **B.** No
- **C.** Maybe
- **D.** Need more info

4. What is the domain of {(0,1),(2,3),(4,5)}?

- **A.** {1,3,5}
- **B.** {0,2,4}
- **C.** {0,1,2,3,4,5}
- **D.** {0,5}

5. What is the range of {(0,1),(2,3),(4,5)}?

- **A.** {1,3,5}
- **B.** {0,2,4}
- **C.** {0,1,2,3,4,5}
- **D.** {0,5}

6. If $g(x) = x^2 + 1$, what is g(3)?

- **A.** 7
- **B.** 10
- **C.** 9
- **D.** 6

7. A babysitter earns 12 birr per hour. The function rule is:

- **A.** f(h) = 12 + h
- **B.** f(h) = 12h
- **C.** f(h) = h/12

- **D.** f(h) = 12 − h

8. Which is read as "f of x"?

- **A.** fx
- **B.** f + x
- **C.** f(x)
- **D.** f/x

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can determine whether a relation is a function.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can use function notation f(x) correctly.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can evaluate a function for a given input.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can identify the domain and range from a set of ordered pairs.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can write a function rule from a real-world description.
Confident ☐ Mostly ☐ Need Review ☐

Unit 2 · Week 5

Domain and Range

Map the boundaries of every function — what goes in and what comes out — as carefully as a Tigrayan farmer maps the edges of his teff field.

LESSON OVERVIEW

Welcome to Week 5. Today we explore the domain and range of functions in depth.

Think of a function like a vending machine. The buttons you can press are the domain. The drinks that can come out are the range.

The domain is the set of all possible inputs. The range is the set of all possible outputs.

From a list of ordered pairs, the domain is the collection of all first numbers — the x-values. The range is the collection of all second numbers — the y-values.

In real life, the domain often has natural limits. The number of hours worked cannot be negative. The number of seats in a theatre cannot exceed its capacity.

Some functions have mathematical restrictions. For example, one over x is not allowed when x equals zero, because division by zero is undefined.

We sometimes describe the domain and range using interval notation. Square brackets include the endpoint. Parentheses exclude it.

For example, the interval from 1 to 5 inclusive is written with square brackets. The interval from 1 to 5 not including the endpoints uses parentheses.

By the end of today, you will identify domain and range from ordered pairs, tables, graphs, and real-life situations.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

`Domain = all inputs (x) · Range = all outputs (y)`

💡 **DID YOU KNOW?**

The word "domain" comes from Latin "dominium" — meaning "ownership" or "realm." A function "owns" or "rules over" its domain — these are the inputs it accepts.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1	**Find domain & range of {(2,4),(3,6),(5,8)}**
Step 1	List all x-values: {2, 3, 5}
Step 2	List all y-values: {4, 6, 8}
ANSWER	**D={2,3,5}, R={4,6,8}**

EXAMPLE 2	**From table x: 1,2,3 · y: 2,4,6**
Step 1	Inputs (left column): 1, 2, 3
Step 2	Outputs (right column): 2, 4, 6
ANSWER	**D={1,2,3}, R={2,4,6}**

EXAMPLE 3	**Graph through (0,1), (2,3), (4,5)**
Step 1	x-coordinates: 0, 2, 4
Step 2	y-coordinates: 1, 3, 5
ANSWER	**D={0,2,4}, R={1,3,5}**

EXAMPLE 4	**Domain for babysitting at 15 birr/hour**
Step 1	Hours cannot be negative
Step 2	Hours cannot be infinite (real limit)
Step 3	Domain: $h \geq 0$

ANSWER **$h \geq 0$ (all non-negative hours)**

EXAMPLE 5 **Domain restriction for $4/(x - 2)$**

Step 1	Denominator cannot be zero
Step 2	$x - 2 \neq 0$
Step 3	Solve: $x \neq 2$

ANSWER **Domain: all reals except $x = 2$**

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 Find domain & range: $\{(1,2), (3,4), (5,6)\}$

2 Find domain & range: $\{(0,0), (2,4), (4,8), (6,12)\}$

3 State the domain restriction for $f(x) = 3/x$

4 State the domain restriction for $f(x) = 5/(x - 3)$

5 State the domain restriction for $f(x) = 2/(x + 7)$

6

Find the domain of $f(x) = \sqrt{x}$ (real numbers).

7

Write in interval notation: all numbers from 1 to 8, both endpoints included.

8

Write in interval notation: all numbers from 0 to 10, endpoints excluded.

9 A babysitter charges 12 birr/hour. State a reasonable domain.

10 Find domain & range: $\{(-2, 5), (0, 7), (3, 11)\}$

11 State the domain restriction for $f(x) = 4/(x^2 - 9)$

12 For $y = \sqrt{(x - 2)}$, what is the domain?

13 Find domain & range from table: $x = 1,2,3,4$; $y = 2,4,6,8$

14 Theater has 200 seats. What is the reasonable domain for ticket sales s?

15 Find domain & range: $\{(-1, 0), (0, 1), (1, 2), (2, 3)\}$

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. Find the domain of {(1,5),(3,7),(6,9)}

- **A.** {5,7,9}
- **B.** {1,3,6}
- **C.** {1,5,3,7,6,9}
- **D.** {1,9}

2. Find the range of {(1,5),(3,7),(6,9)}

- **A.** {5,7,9}
- **B.** {1,3,6}
- **C.** {1,5,3,7,6,9}
- **D.** {5,9}

3. What value must be excluded from the domain of 3/x?

- **A.** 1
- **B.** -1
- **C.** 0
- **D.** 3

4. For f(x) = 5/(x−4), what value is excluded?

- **A.** x = 0
- **B.** x = 4
- **C.** x = 5
- **D.** x = -4

5. Hours worked in a week — what is a reasonable domain?

- **A.** All real numbers
- **B.** $h \geq 0$
- **C.** $h \leq 0$
- **D.** h = 24

6. Interval [2, 8] means:

- **A.** Only 2 and 8
- **B.** 2 to 8, both excluded
- **C.** 2 to 8, both included
- **D.** Less than 2 or greater than 8

7. Interval (2, 8) means:

- **A.** Only 2 and 8
- **B.** 2 to 8, both excluded
- **C.** 2 to 8, both included

- **D.** Less than 2 or greater than 8

8. Domain of $\sqrt{x}$ (real values only):

- **A.** All real numbers
- **B.** $x \geq 0$
- **C.** $x \leq 0$
- **D.** $x \neq 0$

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can identify the domain and range from ordered pairs, tables, and graphs.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can determine restrictions on the domain (where the denominator is zero, or under a square root).
Confident ☐ Mostly ☐ Need Review ☐

✓ I can express intervals using interval notation.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can determine reasonable domains in real-world contexts.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can apply domain and range to real-life situations.
Confident ☐ Mostly ☐ Need Review ☐

Unit 2 · Week 6

Types of Functions

Meet the family of functions — straight, curved, V-shaped, and explosive — each one modelling a different rhythm of the world.

LESSON OVERVIEW

Welcome to Week 6. Today we meet the great family of functions.

Just as Ethiopian music has many rhythms — the steady krar, the rolling masinko, the explosive begena — mathematics has many function types.

A linear function has a constant rate of change. Its graph is a straight line. The form is f of x equals m x plus b.

A quadratic function has its variable squared. Its graph is a U-shaped curve called a parabola. The form is f of x equals a x squared plus b x plus c.

An absolute value function uses absolute value bars. Its graph forms a V-shape, because absolute value always gives a non-negative result.

An exponential function has its variable in the exponent. Its graph grows or decays rapidly. The form is f of x equals a times b to the x power.

A piecewise function uses different rules over different intervals. We will study these in detail next week.

Each function type models a different kind of pattern. Linear models steady wages. Quadratic models projectile motion. Exponential models population growth.

By the end of today, you will identify each function type from its equation, table, or graph.

KEY CONCEPTS & DID YOU KNOW

★ KEY FORMULA

Linear: $mx+b$ · Quadratic: ax^2+bx+c · Absolute: $|x|$ · Exponential: a^{bx}

💡 DID YOU KNOW?

The parabolic shape of quadratic functions appears in nature — the path of water sprayed from a fountain, the cables of a suspension bridge, even the *way a coffee bean falls when dropped.*

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1	**Identify: $f(x) = 5x - 2$**
Step 1	Variable x has power 1
Step 2	No x^2, no $\|x\|$, no exponent
Step 3	Constant rate of change
ANSWER	**Linear Function**

EXAMPLE 2	**Identify: $f(x) = x^2 + 4x + 1$**
Step 1	Highest power of x is 2
Step 2	Graph will be a parabola
ANSWER	**Quadratic Function**

EXAMPLE 3	**Identify: $f(x) = \|x + 2\|$**
Step 1	Contains absolute value bars
Step 2	Graph is V-shaped
ANSWER	**Absolute Value Function**

EXAMPLE 4	**Identify: $f(x) = 3(2)^x$**
Step 1	Variable x is in the exponent
Step 2	Pattern of repeated multiplication

ANSWER **Exponential Function**

EXAMPLE 5 **A babysitter earns 14 birr per hour. What type?**

Step 1	Constant rate: 14 birr per hour
Step 2	Rule: $P(h) = 14h$
Step 3	Steady rate → straight line

ANSWER **Linear Function**

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 Identify: $f(x) = 5x + 9$

2 Identify: $f(x) = x^2 - 4$

3 Identify: $f(x) = |x - 3|$

4 Identify: $f(x) = 2(3)^x$

5 Identify: $f(x) = x^2 + 5x + 6$

6

A bank account doubles every year. What function type?

7

A taxi charges 6 birr base + 2 birr per km. What function type?

8

Identify: $f(x) = |x|$

9

Identify: $f(x) = -3x^2 + 1$

10

Identify: f(x) = 100(0.5)^x

11

A ball thrown into the air follows what function type?

12

A population doubles every 5 years. What function type?

13

Identify: $f(x) = 7x - 2$

14

Identify: $f(x) = |x + 4| - 2$

15

Identify: $f(x) = (1/2)^x$

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. Which is a linear function?
- **A.** $f(x) = x^2 + 1$
- **B.** $f(x) = 3x + 7$
- **C.** $f(x) = 2^x$
- **D.** $f(x) = |x|$

2. Which is a quadratic function?
- **A.** $f(x) = 5x$
- **B.** $f(x) = x^2 - 9$
- **C.** $f(x) = |x|$
- **D.** $f(x) = 4^x$

3. Which graph is V-shaped?
- **A.** Linear
- **B.** Quadratic
- **C.** Absolute value
- **D.** Exponential

4. Which has rapid growth or decay?
- **A.** Linear
- **B.** Quadratic
- **C.** Absolute value
- **D.** Exponential

5. A bank account doubles every year. What type?
- **A.** Linear
- **B.** Quadratic
- **C.** Exponential
- **D.** Absolute value

6. A taxi charges 6 birr plus 2 birr per mile. The function:
- **A.** $f(x) = 2x + 6$
- **B.** $f(x) = 6x^2$
- **C.** $f(x) = 6(2)^x$
- **D.** $f(x) = |x - 6|$

7. Which function has a parabola graph?
- **A.** Linear
- **B.** Quadratic
- **C.** Absolute value

- **D.** Constant

8. $f(x) = x^2$ describes:

- **A.** Linear
- **B.** Quadratic
- **C.** Cubic
- **D.** Exponential

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can identify linear, quadratic, absolute value, and exponential functions from their equations.

Confident ☐ Mostly ☐ Need Review ☐

✓ I can recognize each function type from its graph shape.

Confident ☐ Mostly ☐ Need Review ☐

✓ I can choose the appropriate function type to model a real-world situation.

Confident ☐ Mostly ☐ Need Review ☐

✓ I can compare growth rates of linear and exponential functions.

Confident ☐ Mostly ☐ Need Review ☐

✓ I can describe key features of each function family.

Confident ☐ Mostly ☐ Need Review ☐

Unit 2 · Week 7

Piecewise Functions & Applications

Some situations need more than one rule — like Ethiopian taxi pricing, where the first kilometre costs one rate and every kilometre after costs another.

LESSON OVERVIEW

Welcome to Week 7. This week we explore piecewise functions — functions made of multiple rules.

Real life rarely follows just one rule. A taxi might charge a flat fee for the first kilometre and then a different rate for each kilometre after. That is a piecewise function.

A piecewise function uses different formulas over different intervals of the domain.

We write it with a large curly brace, with each rule next to its condition. For example: f of x equals x plus two when x is less than one, and f of x equals three x when x is greater than or equal to one.

To evaluate a piecewise function, first check which condition the input satisfies. Then apply the matching rule.

When graphing piecewise functions, we use open circles for strict inequalities — less than or greater than. We use closed circles when the endpoint is included — less than or equal to, or greater than or equal to.

A continuous graph has no breaks. A discontinuous graph has jumps, gaps, or holes.

A step function is a piecewise function whose graph looks like a staircase — used for shipping rates, parking fees, and tax brackets.

By the end of today, you will evaluate, write, and graph piecewise functions for real-life situations.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

Choose the rule whose condition matches your input

💡 **DID YOU KNOW?**

Ethiopia's graduated income tax is a piecewise function. Income up to 600 birr is taxed at 0 percent. Income from 600 to 1,650 is taxed at 10 percent. Each bracket has its own rule — exactly like a piecewise function.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1 Evaluate piecewise: $f(x) = \{2x+1 \text{ if } x<3;\ x^2 \text{ if } x\geq3\}$ at $x=2$

Step 1	Check: $2 < 3$ ✓
Step 2	Use rule $2x + 1$
Step 3	$f(2) = 2(2) + 1 = 5$

ANSWER $f(2) = 5$

EXAMPLE 2 Same function at $x = 4$

Step 1	Check: $4 \geq 3$ ✓
Step 2	Use rule x^2
Step 3	$f(4) = 4^2 = 16$

ANSWER $f(4) = 16$

EXAMPLE 3 Write piecewise: gym charges 20 birr for ≤10 visits, +3 per visit over 10

Step 1	For $v \leq 10$: $C = 20$
Step 2	For $v > 10$: $C = 20 + 3(v - 10)$
Step 3	Combine into one piecewise rule

ANSWER $C(v) = 20$ if $v\leq10$, $20+3(v-10)$ if $v>10$

EXAMPLE 4 Graph step function: $f(x) = 1$ if $0\leq x<2$, 3 if $2\leq x<4$

Step 1	Horizontal line at y=1 for [0, 2)
Step 2	Closed circle at (0,1), open at (2,1)
Step 3	Horizontal line at y=3 for [2, 4)

ANSWER **Two horizontal segments at y=1 and y=3**

EXAMPLE 5 Parking: 5 birr first hour, 2 birr each hour after

Step 1	For $h = 1$: $P = 5$
Step 2	For $h > 1$: $P = 5 + 2(h - 1)$
Step 3	Write as piecewise

ANSWER **$P(h) = 5$ if $h=1$, $5+2(h-1)$ if $h>1$**

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 For $f(x) = \{x + 1 \text{ if } x<2;\ 2x \text{ if } x\geq 2\}$, find $f(1)$.

2 For the same f, find $f(5)$.

3 For $f(x) = \{3 \text{ if } x<0;\ x+2 \text{ if } x\geq 0\}$, find $f(-3)$.

4 For the same f, find $f(4)$.

5 On a graph, what does an open circle mean?

6

On a graph, what does a closed (filled) circle mean?

7

A gym charges 20 birr for up to 10 visits, plus 3 birr per visit over 10. Write the piecewise rule for v visits.

8

A parking garage charges 5 birr for the first hour, then 2 birr per hour after. Write the piecewise function for h hours.

9 For $f(x) = \{x^2 \text{ if } x<0;\ 2x \text{ if } x\geq 0\}$, find $f(-3)$.

10 For the same f, find f(2).

11 What is a step function? Give a real-world example.

12 Define "continuous" vs "discontinuous" functions.

13 For $f(x) = \{2x+1 \text{ if } x<1;\ x^2 \text{ if } x\geq 1\}$, find $f(0)$ and $f(3)$.

14 Ethiopia's graduated income tax is a piecewise function. Why?

15

Sketch the graph: $f(x) = 1$ for $0 \leq x < 2$, $f(x) = 3$ for $2 \leq x < 4$.

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. For $f(x) = \{x+1 \text{ if } x<2;\ 2x \text{ if } x\geq 2\}$, find $f(1)$

- **A.** 2
- **B.** 3
- **C.** 1
- **D.** 0

2. For the same $f(x)$, find $f(3)$

- **A.** 4
- **B.** 5
- **C.** 6
- **D.** 7

3. For $f(x) = \{3 \text{ if } x<0;\ x+1 \text{ if } x\geq 0\}$, find $f(-5)$

- **A.** −4
- **B.** 3
- **C.** 5
- **D.** 0

4. Which symbol on a graph means "not included"?

- **A.** Closed circle
- **B.** Open circle
- **C.** Solid line
- **D.** Dashed line

5. Which describes a graph with no breaks?

- **A.** Discontinuous
- **B.** Step
- **C.** Continuous
- **D.** Discrete

6. A step function graph looks like:

- **A.** A straight line
- **B.** A parabola
- **C.** A V
- **D.** A staircase

7. Why are piecewise functions useful in real life?

- **A.** They look complicated
- **B.** One rule cannot describe all situations
- **C.** They are always continuous

- **D.** They have no domain

8. For f(x) = {x^2 if $x<0$; $2x$ if $x\geq 0$}, find f(−3)

- **A.** 9
- **B.** −6
- **C.** 6
- **D.** −9

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can evaluate a piecewise function for any given input.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can graph piecewise functions using open and closed circles.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can write piecewise functions from real-world descriptions.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can identify whether a graph is continuous or discontinuous.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can apply piecewise functions to pricing, taxes, and similar models.
Confident ☐ Mostly ☐ Need Review ☐

Unit 2 Answer Key

Compare your answers below. If you missed a question, review the corresponding lesson section.

Week 4: Function Notation

Q#	Answer	Explanation
1	**B**	Substitute 5 for x: f(5) = 2(5) + 3 = 10 + 3 = 13.
2	**A**	Every input has exactly one output, so it is a function.
3	**B**	Input 2 maps to both 5 and 7, so it is NOT a function.
4	**B**	Domain = all inputs (x-values) = {0, 2, 4}.
5	**A**	Range = all outputs (y-values) = {1, 3, 5}.
6	**B**	$g(3) = 3^2 + 1 = 9 + 1 = 10$.
7	**B**	Total earnings = rate × hours = 12h.
8	**C**	f(x) is read "f of x" and represents the function f acting on input x.

Week 5: Domain and Range

Q#	Answer	Explanation
1	**B**	Domain = all x-values = {1, 3, 6}.
2	**A**	Range = all y-values = {5, 7, 9}.
3	**C**	Division by zero is undefined, so x ≠ 0.
4	**B**	Set denominator ≠ 0: x − 4 ≠ 0, so x ≠ 4.
5	**B**	Hours cannot be negative — domain is h ≥ 0.
6	**C**	Square brackets include the endpoints.
7	**B**	Round parentheses exclude the endpoints.
8	**B**	You cannot take the square root of a negative number (in real numbers), so x ≥ 0.

Week 6: Types of Functions

Q#	Answer	Explanation
1	B	$f(x) = 3x + 7$ has x to the first power → linear.
2	B	The highest power is 2 (x^2) → quadratic.
3	C	Absolute value functions produce V-shaped graphs.
4	D	Exponential functions grow (or decay) very rapidly.
5	C	Repeated multiplication (doubling) is exponential.
6	A	Fixed fee + per-mile rate is linear: $6 + 2x$.
7	B	Quadratic functions produce parabolas (U-shaped curves).
8	B	x^2 has degree 2 → quadratic.

Week 7: Piecewise Functions & Applications

Q#	Answer	Explanation
1	A	Since $1 < 2$, use $x+1$. $f(1) = 1+1 = 2$.
2	C	Since $3 \geq 2$, use $2x$. $f(3) = 2(3) = 6$.
3	B	$-5 < 0$, so use the constant rule 3. $f(-5) = 3$.
4	B	Open circle means the point is NOT included (strict inequality).
5	C	A continuous graph has no breaks, jumps, or holes.
6	D	Step functions produce graphs shaped like steps or staircases.
7	B	Many real situations (pricing, taxes, shipping) need different rules for different ranges.
8	A	$-3 < 0$, use x^2. $f(-3) = (-3)^2 = 9$.

UNIT 3

Polynomials

WEEKS COVERED

Week 8: Polynomial Operations
Week 9: Dividing Polynomials
Week 10: Factoring Techniques
Week 11: Advanced Factoring

Unit 3 · Week 8

Polynomial Operations

Add, subtract, and multiply polynomial expressions — the building blocks of every algebraic curve, from a parabola to a coffee-bean trajectory.

LESSON OVERVIEW

Welcome to Unit 3. This week we begin our study of polynomials — one of the most important families of expressions in algebra.

A polynomial is an expression made of variables, coefficients, and whole-number exponents. For example, three x squared plus five x minus one is a polynomial.

Each piece separated by a plus or minus sign is called a term. The number in front of the variable is called the coefficient. The exponent on the variable is called the degree of that term.

The degree of the whole polynomial is the highest exponent that appears. A polynomial of degree one is linear. Degree two is quadratic. Degree three is cubic.

A polynomial with one term is a monomial. With two terms, a binomial. With three terms, a trinomial.

To add polynomials, we simply combine like terms — terms that have the same variable and same exponent.

To subtract, we distribute the negative sign through the second polynomial, then combine like terms.

To multiply, we use the distributive property. For two binomials, we often use the FOIL pattern: First, Outer, Inner, Last.

There are also special products to memorize: the square of a binomial equals a squared plus 2ab plus b squared.

By the end of today, you will add, subtract, and multiply polynomials with confidence.

KEY CONCEPTS & DID YOU KNOW

★ KEY FORMULA

$(a + b)^2 = a^2 + 2ab + b^2 \cdot (a - b)(a + b) = a^2 - b^2$

💡 **DID YOU KNOW?**

Polynomials describe the curves of suspension bridges, the trajectory of an Ethiopian javelin throw, and even the shape of the dome on Holy Trinity Cathedral in Addis Ababa.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1 **Add $(3x^2 + 2x + 5) + (x^2 - 4x + 1)$**	
Step 1	Combine x^2 terms: $3x^2 + x^2 = 4x^2$
Step 2	Combine x terms: $2x - 4x = -2x$
Step 3	Combine constants: $5 + 1 = 6$
ANSWER	**$4x^2 - 2x + 6$**

EXAMPLE 2 **Subtract $(5x^2 + 3x - 2) - (2x^2 - x + 4)$**	
Step 1	Distribute the negative: $5x^2 + 3x - 2 - 2x^2 + x - 4$
Step 2	Combine x^2: $5x^2 - 2x^2 = 3x^2$
Step 3	Combine x: $3x + x = 4x$
Step 4	Combine constants: $-2 - 4 = -6$
ANSWER	**$3x^2 + 4x - 6$**

EXAMPLE 3 **Multiply $3x(x^2 + 2x - 5)$**	
Step 1	Distribute 3x to each term
Step 2	$3x \cdot x^2 = 3x^3$
Step 3	$3x \cdot 2x = 6x^2$
Step 4	$3x \cdot (-5) = -15x$
ANSWER	**$3x^3 + 6x^2 - 15x$**

EXAMPLE 4 Multiply $(x + 3)(x + 5)$ using FOIL

Step 1	F: $x \cdot x = x^2$
Step 2	O: $x \cdot 5 = 5x$
Step 3	I: $3 \cdot x = 3x$
Step 4	L: $3 \cdot 5 = 15$
Step 5	Combine: $x^2 + 5x + 3x + 15$

ANSWER $x^2 + 8x + 15$

EXAMPLE 5 Special product: $(x + 4)^2$

Step 1	Use $(a + b)^2 = a^2 + 2ab + b^2$
Step 2	$a = x$, $b = 4$
Step 3	Result: $x^2 + 2(x)(4) + 16$

ANSWER $x^2 + 8x + 16$

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 Add: $(2x^2 + 3x + 1) + (x^2 - x + 4)$

2 Add: $(5x^2 + 2x - 3) + (3x^2 + 4x + 1)$

3 Subtract: $(4x^2 + 5x - 7) - (x^2 + 2x + 3)$

4 Subtract: $(7x^2 - 3x + 8) - (2x^2 + 5x - 4)$

5 Multiply: $2x(x^2 - 3x + 6)$

6

Multiply: $4x(2x^2 + x - 5)$

7

Multiply: $(x + 2)(x + 7)$

8

Multiply: $(x + 3)(x - 5)$

9 Multiply: $(x - 4)(x - 2)$

10 Expand: $(x + 5)^2$

11 Expand: $(x - 3)^2$

12 Expand: $(x + 6)(x - 6)$ [difference of squares]

13 Classify by terms and degree: $5x^2 - 2x + 1$

14 Classify: $7x^3$

15

Classify: $4x - 9$

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. Add: $(2x^2 + 3x + 1) + (x^2 + 4x - 2)$

- **A.** $3x^2 + 7x - 1$
- **B.** $3x^2 - x + 3$
- **C.** $2x^2 + 7x + 1$
- **D.** $3x^2 + 7x + 1$

2. Subtract: $(5x^2 + 2x) - (x^2 + 3x)$

- **A.** $4x^2 + 5x$
- **B.** $4x^2 - x$
- **C.** $6x^2 - x$
- **D.** $4x^2 + x$

3. Multiply: $4x(x + 3)$

- **A.** $4x + 12$
- **B.** $4x^2 + 12$
- **C.** $4x^2 + 12x$
- **D.** $5x + 3$

4. Multiply: $(x + 2)(x + 6)$

- **A.** $x^2 + 8x + 12$
- **B.** $x^2 + 12x + 8$
- **C.** $x^2 + 8$
- **D.** $x^2 + 8x + 8$

5. Expand: $(x - 3)^2$

- **A.** $x^2 - 9$
- **B.** $x^2 + 9$
- **C.** $x^2 - 6x + 9$
- **D.** $x^2 - 3x + 9$

6. Classify by terms: $5x^2 - 2x + 1$

- **A.** Monomial
- **B.** Binomial
- **C.** Trinomial
- **D.** Quadrinomial

7. What is the degree of $x^3 + 2x^2 + 1$?

- **A.** 1
- **B.** 2
- **C.** 3

D. 6

8. $(x + 5)(x - 5)$ equals:

A. $x^2 - 25$
B. $x^2 + 25$
C. $x^2 - 10x + 25$
D. $x^2 - 10x - 25$

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can identify and classify polynomials by degree and number of terms.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can add and subtract polynomials.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can multiply polynomials (monomial × polynomial, FOIL).
Confident ☐ Mostly ☐ Need Review ☐

✓ I can apply special product formulas, $(a+b)^2$ and $(a-b)(a+b)$.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can simplify polynomial expressions in standard form.
Confident ☐ Mostly ☐ Need Review ☐

Unit 3 · Week 9

Dividing Polynomials

Master the art of polynomial division — both the long-division technique and the elegant synthetic shortcut.

LESSON OVERVIEW

Welcome to Week 9. This week, we tackle polynomial division — splitting larger expressions into smaller, simpler ones.

There are three main techniques: dividing by a monomial, polynomial long division, and synthetic division.

When dividing by a monomial, we simply divide each term separately. For example, 6 x cubed plus 9 x squared, divided by 3 x, gives 2 x squared plus 3 x.

Polynomial long division works just like regular long division of numbers. We have a dividend, a divisor, a quotient, and sometimes a remainder.

The steps are: divide the leading terms, multiply, subtract, bring down the next term, and repeat.

Synthetic division is a beautiful shortcut. It only works when dividing by a linear factor of the form x minus c. We use just the coefficients, no variables — making the work much faster.

In synthetic division, we bring down the first coefficient, multiply by c, add to the next coefficient, and continue down the line.

Always remember: if the dividend has a missing term, insert a zero for that coefficient before beginning division.

We can verify any division by multiplying the divisor by the quotient and adding the remainder. The result should match the original dividend.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

Dividend = Divisor × Quotient + Remainder

💡 **DID YOU KNOW?**

Synthetic division was developed in the 1700s as a shortcut for the long, *tedious work of polynomial division — saving mathematicians hours of writing.*

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1	**Divide $(8x^4 + 12x^3) \div 4x$**
Step 1	Divide each term separately
Step 2	$8x^4 \div 4x = 2x^3$
Step 3	$12x^3 \div 4x = 3x^2$
ANSWER	**$2x^3 + 3x^2$**

EXAMPLE 2	**Long divide $(x^2 + 5x + 6) \div (x + 2)$**
Step 1	Divide leading: $x^2 \div x = x$
Step 2	Multiply: $x(x+2) = x^2+2x$
Step 3	Subtract: leaves $3x+6$
Step 4	Divide: $3x \div x = 3$
Step 5	Multiply: $3(x+2) = 3x+6$
Step 6	Subtract: remainder 0
ANSWER	**$x + 3$**

EXAMPLE 3	**Long divide with remainder: $(x^2 + 4) \div (x + 1)$**
Step 1	Insert missing term: $x^2 + 0x + 4$
Step 2	Divide: $x^2 \div x = x$
Step 3	Multiply, subtract: leaves $-x + 4$

Step 4	Divide: $-x \div x = -1$
Step 5	Multiply, subtract: remainder 5

ANSWER $x - 1 + 5/(x+1)$

EXAMPLE 4 **Synthetic divide $(x^3 + 2x^2 - 5x + 6) \div (x - 2)$**

Step 1	Use $c = 2$; coefficients: 1, 2, −5, 6
Step 2	Bring down 1
Step 3	Multiply $1\cdot 2=2$, add to 2: get 4
Step 4	Multiply $4\cdot 2=8$, add to −5: get 3
Step 5	Multiply $3\cdot 2=6$, add to 6: get 12 (remainder)

ANSWER $x^2 + 4x + 3 + 12/(x-2)$

EXAMPLE 5 **Verify: does $(x+3)(x+2) = x^2+5x+6$?**

Step 1	FOIL: $x\cdot x + x\cdot 2 + 3\cdot x + 3\cdot 2$
Step 2	$= x^2 + 2x + 3x + 6$
Step 3	$= x^2 + 5x + 6$ ✓

ANSWER **Yes — division was correct**

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 Divide: $(12x^3 + 18x^2) \div 6x$

2 Divide: $(15x^4 - 10x^3) \div 5x^2$

3 Long divide: $(x^2 + 7x + 10) \div (x + 5)$

4 Long divide: $(x^2 + 5x + 6) \div (x + 2)$

5 Long divide: $(x^2 - 9) \div (x + 3)$

6

Synthetic divide: $(x^3 - 3x^2 + 2x + 8) \div (x - 2)$

7

Synthetic divide: $(x^3 + 2x^2 - 5x + 6) \div (x - 2)$

8

Long divide with remainder: $(x^2 + 1) \div (x + 1)$

9

Verify by multiplication: $(x + 4)(x + 2)$. Does it equal $x^2 + 6x + 8$?

10

Verify: $(x - 3)(x + 7)$. Does it equal $x^2 + 4x - 21$?

11

Divide: $(8x^3 - 4x^2) \div 4x$

12

Long divide: $(x^2 - 4x + 4) \div (x - 2)$

13

Synthetic divide: $(x^3 + x^2 - 4x - 4) \div (x + 1)$

14

Find quotient AND remainder: $(x^2 + 3) \div (x - 1)$

15

Divide: $(20x^5 + 10x^4) \div 5x^2$

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. Divide: $(12x^3 + 18x^2) \div 6x$

- **A.** $2x^2 + 3x$
- **B.** $2x^3 + 3x^2$
- **C.** $6x^2 + 12x$
- **D.** $12x^2 + 18x$

2. Divide: $(x^2 + 7x + 10) \div (x + 5)$

- **A.** $x + 2$
- **B.** $x + 5$
- **C.** $x + 10$
- **D.** $x - 2$

3. In synthetic division for ÷ $(x - 3)$, what value do we use?

- **A.** -3
- **B.** 3
- **C.** x
- **D.** 1/3

4. In synthetic division for ÷ $(x + 4)$, what value do we use?

- **A.** 4
- **B.** -4
- **C.** x
- **D.** 1/4

5. What must you do when the dividend has a missing term?

- **A.** Skip it
- **B.** Insert a zero coefficient
- **C.** Add 1
- **D.** Multiply by x

6. Dividend = Divisor × Quotient + ?

- **A.** Dividend
- **B.** Remainder
- **C.** Coefficient
- **D.** Variable

7. $(6x^2 + 9x) \div 3x$ equals:

- **A.** $2x + 3$
- **B.** $2x^2 + 3$
- **C.** $3x + 9$

D. $6x + 9$

8. What is the divisor in $(x^2+5x+6)\div(x+2)$?

A. $x^2 + 5x + 6$
B. $x + 2$
C. $x + 3$
D. x

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can divide a polynomial by a monomial.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can perform polynomial long division.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can use synthetic division when dividing by a linear factor.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can write the quotient and remainder of a polynomial division.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can verify a division by multiplying back.
Confident ☐ Mostly ☐ Need Review ☐

Unit 3 · Week 10

Factoring Techniques

Reverse multiplication — break polynomials apart to reveal their hidden structure, just as a Tigrayan stonemason finds the natural cleavage in obsidian.

LESSON OVERVIEW

Welcome to Week 10. Today we learn factoring — the reverse of multiplication.

Factoring rewrites an expression as a product of simpler expressions. For example, x squared plus five x plus six factors as the product (x plus 2)(x plus 3).

Why factor? Because factored expressions are easier to solve, simplify, and analyze. Almost every quadratic equation can be solved through factoring.

The first step in any factoring problem is to look for the Greatest Common Factor, or GCF. Pull out the largest common number and variable from every term.

To factor a trinomial of the form x squared plus b x plus c, we find two numbers that multiply to c and add to b.

For example, x squared plus 5 x plus 6: we need two numbers multiplying to 6 and adding to 5. Those are 2 and 3. So the factors are (x plus 2)(x plus 3).

There are also special patterns. The difference of squares: a squared minus b squared equals (a minus b)(a plus b). The perfect square trinomial: a squared plus 2 a b plus b squared equals (a plus b) squared.

When a polynomial has four terms, we often factor by grouping. Group the first two and last two terms, factor each group, and look for a common binomial.

Always factor completely. Check your work by multiplying the factors back to see if you get the original.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

GCF · Trinomial · Difference of Squares (a^2-b^2) · Grouping

💡 **DID YOU KNOW?**

The word "factor" comes from the Latin "facere" — meaning "to make" or "to *do." Factors are the building blocks that "make up" a number or expression.*

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1 Factor using GCF: $8x^3 + 12x^2$

Step 1	GCF of 8 and 12 is 4
Step 2	Common variable: x^2
Step 3	Factor out $4x^2$

ANSWER $4x^2(2x + 3)$

EXAMPLE 2 Factor trinomial: $x^2 + 5x + 6$

Step 1	Find two numbers that multiply to 6 and add to 5
Step 2	Numbers: 2 and 3
Step 3	Write as $(x + 2)(x + 3)$

ANSWER $(x + 2)(x + 3)$

EXAMPLE 3 Difference of squares: $x^2 - 16$

Step 1	Recognize: x^2 and $16 = 4^2$
Step 2	Apply $(a - b)(a + b)$
Step 3	$a = x$, $b = 4$

ANSWER $(x - 4)(x + 4)$

EXAMPLE 4 Perfect square: $x^2 + 10x + 25$

Step 1	Recognize $25 = 5^2$
Step 2	Middle term: $2(x)(5) = 10x$ ✓
Step 3	Use $(a + b)^2$

ANSWER $(x + 5)^2$

EXAMPLE 5 **Factor by grouping: $x^3 + 2x^2 + 3x + 6$**

Step 1	Group: $(x^3+2x^2) + (3x+6)$
Step 2	Factor each: $x^2(x+2) + 3(x+2)$
Step 3	Factor common binomial $(x+2)$

ANSWER $(x + 2)(x^2 + 3)$

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 Factor using GCF: $10x^2 + 15x$

2 Factor using GCF: $8x^3 + 12x^2$

3 Factor using GCF: $6x^2 - 9x$

4 Factor: $x^2 + 9x + 20$

5 Factor: $x^2 + 7x + 12$

6

Factor: $x^2 - 6x + 8$

7

Factor: $x^2 + 5x + 6$

8

Factor: $x^2 - 7x + 10$

9 Factor: $x^2 - 49$ (difference of squares)

10 Factor: $x^2 - 64$

11 Factor: $x^2 + 6x + 9$ (perfect square)

12 Factor: $x^2 - 10x + 25$

13 Factor by grouping: $x^3 + 4x^2 + 2x + 8$

14 Factor by grouping: $x^3 + 2x^2 + 3x + 6$

15

Factor: $x^2 - 16$

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. Factor using GCF: $10x^2 + 15x$

- **A.** $5(2x^2 + 3x)$
- **B.** $5x(2x + 3)$
- **C.** $10x(x + 15)$
- **D.** $x(10x + 15)$

2. Factor: $x^2 + 9x + 20$

- **A.** $(x+4)(x+5)$
- **B.** $(x+2)(x+10)$
- **C.** $(x+4)(x-5)$
- **D.** $(x+1)(x+20)$

3. Factor: $x^2 - 49$

- **A.** $(x-7)^2$
- **B.** $(x+7)^2$
- **C.** $(x-7)(x+7)$
- **D.** $(x-49)(x+1)$

4. Factor: $x^2 + 6x + 9$

- **A.** $(x+3)^2$
- **B.** $(x+9)(x+1)$
- **C.** $(x-3)^2$
- **D.** $(x+3)(x+6)$

5. Factor by grouping: $x^3 + 4x^2 + 2x + 8$

- **A.** $(x+4)(x^2+2)$
- **B.** $(x+2)(x^2+4)$
- **C.** $(x^2+4)(x+2)$
- **D.** $(x+4)(x+2)$

6. Factor: $x^2 - 25$

- **A.** $(x-5)^2$
- **B.** $(x-5)(x+5)$
- **C.** $(x+5)^2$
- **D.** Cannot factor

7. Factor: $x^2 + 7x + 10$

- **A.** $(x+5)(x+2)$
- **B.** $(x+1)(x+10)$
- **C.** $(x+5)(x-2)$

- **D.** $(x-5)(x+2)$

8. What is the GCF of $6x^2$ and $9x$?

- **A.** 3
- **B.** x
- **C.** $3x$
- **D.** $6x$

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can factor out the greatest common factor (GCF).
Confident ☐ Mostly ☐ Need Review ☐

✓ I can factor quadratic trinomials of the form $x^2 + bx + c$.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can factor a difference of squares.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can recognize and factor perfect square trinomials.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can factor four-term polynomials by grouping.
Confident ☐ Mostly ☐ Need Review ☐

Unit 3 · Week 11

Advanced Factoring

Take factoring to the next level — leading coefficients, cubes, substitution, and the elegant detection of "prime" polynomials.

LESSON OVERVIEW

Welcome to Week 11. We now extend our factoring tools to handle harder cases.

When the leading coefficient of a quadratic trinomial is greater than one, we use the "AC method." Multiply the first and last coefficients, then find two numbers that multiply to that product and add to the middle coefficient.

For example, 2 x squared plus 7 x plus 3. Multiply 2 times 3 to get 6. We need two numbers multiplying to 6 and adding to 7. Those are 6 and 1. Rewrite, then factor by grouping.

There are also formulas for cubes. The difference of cubes: a cubed minus b cubed equals (a minus b)(a squared plus a b plus b squared). The sum of cubes: a cubed plus b cubed equals (a plus b)(a squared minus a b plus b squared).

For higher-degree polynomials, we sometimes use substitution. For example, x to the fourth minus 5 x squared plus 4: let u equal x squared, factor in terms of u, then substitute back.

Always factor completely. After applying one method, check whether the resulting factors can be factored further.

Some polynomials cannot be factored using integers. These are called prime polynomials. An example is x squared plus x plus 1 — no integer factors work.

A complete factoring strategy: first GCF, then look for special patterns, then trinomial methods, then grouping. Always verify by multiplication.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

$$a^3-b^3 = (a-b)(a^2+ab+b^2) \cdot a^3+b^3 = (a+b)(a^2-ab+b^2)$$

💡 **DID YOU KNOW?**

The sum and difference of cubes formulas were known to Italian mathematicians in the 1500s — particularly Gerolamo Cardano, who used *them to solve cubic equations centuries before modern algebra was formalized.*

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1 AC method: $2x^2 + 7x + 3$

Step 1	Multiply $2 \cdot 3 = 6$
Step 2	Find numbers: 6 and 1 (multiply to 6, add to 7)
Step 3	Rewrite: $2x^2 + 6x + x + 3$
Step 4	Group: $2x(x+3) + 1(x+3)$
Step 5	Factor common binomial

ANSWER $(2x + 1)(x + 3)$

EXAMPLE 2 Difference of cubes: $x^3 - 27$

Step 1	Recognize $27 = 3^3$
Step 2	Apply $a^3 - b^3 = (a-b)(a^2+ab+b^2)$
Step 3	$a = x$, $b = 3$

ANSWER $(x - 3)(x^2 + 3x + 9)$

EXAMPLE 3 Sum of cubes: $x^3 + 64$

Step 1	Recognize $64 = 4^3$
Step 2	Apply $a^3 + b^3 = (a+b)(a^2-ab+b^2)$
Step 3	$a = x$, $b = 4$

ANSWER $(x + 4)(x^2 - 4x + 16)$

EXAMPLE 4 Substitution: $x^4 - 5x^2 + 4$

Step 1	Let $u = x^2$
Step 2	Factor $u^2 - 5u + 4 = (u-1)(u-4)$
Step 3	Replace u: $(x^2-1)(x^2-4)$
Step 4	Factor further (both difference of squares)

ANSWER **$(x-1)(x+1)(x-2)(x+2)$**

EXAMPLE 5 Identify prime: $x^2 + x + 1$

Step 1	Look for integers multiplying to 1, adding to 1
Step 2	No such integers exist
Step 3	Cannot be factored over integers

ANSWER **Prime polynomial**

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 Factor: $3x^2 + 10x + 3$

2 Factor: $2x^2 + 5x + 2$

3 Factor: $2x^2 + 7x + 3$

4 Factor (difference of cubes): $x^3 - 64$

5 Factor (difference of cubes): $x^3 - 27$

6

Factor (sum of cubes): $x^3 + 125$

7

Factor (sum of cubes): $x^3 + 8$

8

Factor completely: $x^4 - 13x^2 + 36$

9 Determine if prime: $x^2 + 2x + 2$

10 Determine if prime: $x^2 + x + 1$

11 Factor: $5x^2 + 11x + 2$

12 Factor: $x^3 - 1$ (difference of cubes)

13 Factor completely: $2x^4 - 8x^2$

14 Factor: $4x^2 - 25$ (difference of squares)

15

Factor: $3x^3 + 24$ (sum of cubes after GCF)

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. Factor: $3x^2 + 10x + 3$

- **A.** $(3x+1)(x+3)$
- **B.** $(3x+3)(x+1)$
- **C.** $(x+3)(x+1)$
- **D.** $(3x+10)(x+3)$

2. Factor: $x^3 - 64$

- **A.** $(x-4)^3$
- **B.** $(x-4)(x^2+4x+16)$
- **C.** $(x-4)(x^2-4x+16)$
- **D.** $(x+4)(x^2-4x+16)$

3. Factor: $x^3 + 125$

- **A.** $(x+5)^3$
- **B.** $(x+5)(x^2-5x+25)$
- **C.** $(x+5)(x^2+5x+25)$
- **D.** $(x-5)(x^2+5x+25)$

4. Factor completely: $x^4 - 13x^2 + 36$

- **A.** $(x^2-4)(x^2-9)$
- **B.** $(x-2)(x+2)(x-3)(x+3)$
- **C.** $(x^2-4)(x^2-36)$
- **D.** Both A and B

5. Is $x^2 + 2x + 2$ prime?

- **A.** Yes
- **B.** No, equals $(x+1)^2$
- **C.** No, equals $(x+2)(x+1)$
- **D.** No, equals $(x-2)(x-1)$

6. The formula $a^3 - b^3$ equals:

- **A.** $(a-b)(a^2+ab+b^2)$
- **B.** $(a-b)(a^2-ab+b^2)$
- **C.** $(a+b)(a^2-ab+b^2)$
- **D.** $(a-b)^3$

7. Factor: $2x^2 + 5x + 2$

- **A.** $(2x+1)(x+2)$
- **B.** $(2x+2)(x+1)$
- **C.** $(x+1)(2x+2)$

- **D.** (x+2)(2x+1)

8. First step in any factoring problem:
- **A.** Try grouping
- **B.** Look for GCF
- **C.** Use cubes formula
- **D.** Try substitution

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can factor trinomials with leading coefficients greater than 1 (AC method).
Confident ☐ Mostly ☐ Need Review ☐

✓ I can factor a sum or difference of cubes.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can use substitution to factor higher-degree polynomials.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can factor polynomials completely using multiple methods.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can recognize when a polynomial is prime over the integers.
Confident ☐ Mostly ☐ Need Review ☐

Unit 3 Answer Key

Compare your answers below. If you missed a question, review the corresponding lesson section.

Week 8: Polynomial Operations

Q#	Answer	Explanation
1	A	Combine like terms: $2x^2+x^2=3x^2$, $3x+4x=7x$, $1-2=-1$.
2	B	Distribute negative: $5x^2+2x-x^2-3x = 4x^2-x$.
3	C	Distribute: $4x\cdot x + 4x\cdot 3 = 4x^2 + 12x$.
4	A	FOIL: $x^2 + 6x + 2x + 12 = x^2 + 8x + 12$.
5	C	$(a-b)^2 = a^2-2ab+b^2 \rightarrow x^2 - 6x + 9$.
6	C	Three terms = trinomial.
7	C	The highest exponent is 3.
8	A	Difference of squares: $(a+b)(a-b) = a^2 - b^2 = x^2 - 25$.

Week 9: Dividing Polynomials

Q#	Answer	Explanation
1	A	Each term: $12x^3/6x = 2x^2$, $18x^2/6x = 3x$.
2	A	Long division gives quotient $x + 2$ with no remainder.
3	B	For divisor $(x - c)$, use c. Here $c = 3$.
4	B	Rewrite $(x + 4)$ as $(x - (-4))$, so $c = -4$.
5	B	Insert 0 as the placeholder coefficient (e.g., $x^2 + 4$ becomes $x^2 + 0x + 4$).
6	B	The check formula: Dividend = Divisor × Quotient + Remainder.
7	A	$6x^2/3x = 2x$, $9x/3x = 3$. Answer: $2x + 3$.
8	B	The divisor is the polynomial we divide BY: $(x + 2)$.

Week 10: Factoring Techniques

Q#	Answer	Explanation
1	B	GCF is 5x: 5x(2x + 3).
2	A	Two numbers: 4·5=20, 4+5=9. Answer: (x+4)(x+5).
3	C	Difference of squares: 49 = 7^2. (x−7)(x+7).
4	A	Perfect square: 9 = 3^2, 2(3)x = 6x. $(x+3)^2$.
5	A	Group: $x^2(x+4) + 2(x+4) = (x+4)(x^2+2)$.
6	B	Difference of squares: 25 = 5^2. (x−5)(x+5).
7	A	Two numbers: 5·2=10, 5+2=7. (x+5)(x+2).
8	C	Number GCF=3, variable GCF=x. Combined: 3x.

Week 11: Advanced Factoring

Q#	Answer	Explanation
1	A	AC: 3·3=9, find 9 and 1. Rewrite, group: (3x+1)(x+3).
2	B	Difference of cubes: $64=4^3$. $(x-4)(x^2+4x+16)$.
3	B	Sum of cubes: $125=5^3$. $(x+5)(x^2-5x+25)$.
4	D	First $(x^2-4)(x^2-9)$, then each is a difference of squares: (x−2)(x+2)(x−3)(x+3). Both forms are correct, B is complete.
5	A	No integers multiply to 2 and add to 2. Prime over integers.
6	A	Difference of cubes: $(a-b)(a^2+ab+b^2)$.
7	A	AC: 2·2=4, find 4 and 1: rewrite as $2x^2+4x+x+2$, group: (2x+1)(x+2). Note: this is equivalent to D.
8	B	Always pull out the Greatest Common Factor first.

UNIT 4

Rational Expressions & Equations

WEEKS COVERED

Week 12: Simplifying Rational Expressions
Week 13: Multiplying & Dividing Rational Expressions
Week 14: Solving Rational Equations
Week 15: Applications of Rational Equations

Unit 4 · Week 12

Simplifying Rational Expressions

Rational expressions are fractions made of polynomials — like dividing the bountiful injera plate fairly among many guests at an Ethiopian feast.

LESSON OVERVIEW

Welcome to Unit 4. This week we begin our study of rational expressions — fractions where the numerator and denominator are polynomials.

A rational expression looks like a regular fraction, but with variables. For example, x squared minus nine, divided by x squared plus three x.

The most important rule about any fraction is that the denominator cannot equal zero. Division by zero is undefined.

So before we simplify, we identify the restrictions — the values that would make the denominator zero. Those values are excluded from the domain.

To simplify a rational expression, we factor both the numerator and denominator, then cancel common factors.

Critical rule: we cancel only common factors, not common terms. A factor multiplies the entire expression. A term is separated by plus or minus.

For example, in the fraction (x minus 3)(x plus 3) divided by x times (x plus 3), the factor (x plus 3) appears in both. We can cancel it.

After cancelling, the simplified form is (x minus 3) divided by x, with restrictions that x cannot equal 0 or negative 3.

Always state restrictions even after simplifying. The original expression had those limits, and so does the simplified form.

By the end of today, you will simplify rational expressions and properly state all restrictions.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

Factor · Cancel common FACTORS only · State restrictions (denom ≠ 0)

💡 **DID YOU KNOW?**

Rational expressions appear everywhere in engineering — from the formulas that describe coffee brewing temperatures to the equations of fluid flow in Ethiopian hydroelectric dams.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1	**Simplify $(x^2 - 9)/(x^2 + 3x)$**
Step 1	Factor numerator: $(x-3)(x+3)$
Step 2	Factor denominator: $x(x+3)$
Step 3	Restrictions: $x \neq 0, -3$
Step 4	Cancel $(x+3)$ common factor
ANSWER	**$(x - 3)/x, x \neq 0, -3$**

EXAMPLE 2	**Simplify $(2x^2 + 4x)/4x$**
Step 1	Factor numerator: $2x(x+2)$
Step 2	Restriction: $x \neq 0$
Step 3	Cancel 2x with 4x: leaves /2
ANSWER	**$(x + 2)/2, x \neq 0$**

EXAMPLE 3	**Multiply: $x/(x+2) \cdot (x+2)/(x-3)$**
Step 1	Identify common factor: $(x+2)$
Step 2	Cancel it
Step 3	Restrictions: $x \neq -2, 3$
ANSWER	**$x/(x - 3), x \neq -2, 3$**

EXAMPLE 4 **Divide: $x/(x-1) \div (x+2)/x$**

Step 1	Multiply by reciprocal
Step 2	$x/(x-1) \cdot x/(x+2)$
Step 3	Multiply numerators and denominators

ANSWER **$x^2/((x-1)(x+2)),\ x \neq 1, -2, 0$**

EXAMPLE 5 **Simplify $(x^2 - 5x + 6)/(x^2 - x - 6)$**

Step 1	Factor numerator: $(x-2)(x-3)$
Step 2	Factor denominator: $(x-3)(x+2)$
Step 3	Cancel $(x-3)$
Step 4	Restrictions: $x \neq 3, -2$

ANSWER **$(x - 2)/(x + 2),\ x \neq 3, -2$**

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 What value(s) must be excluded from $x/(x - 5)$?

2 What value(s) must be excluded from $3/(x + 2)$?

3 Simplify: $(3x^2)/(6x)$. State restrictions.

4 Simplify: $(x^2 - 9)/(x + 3)$. State restrictions.

5 Simplify: $(x^2 + 5x + 6)/(x + 2)$. State restrictions.

6

Simplify: $(x^2 - 16)/(x - 4)$

7

Simplify: $(2x + 4)/2$

8

Simplify: $(x^2 - 5x + 6)/(x^2 - x - 6)$. State restrictions.

9

Simplify: $(4x + 8)/4$

10

State the restrictions for $3/(x^2 - 4)$.

11

Simplify: $(x^2 - 1)/(x - 1)$. State restrictions.

12

Simplify: $x/(x+3) \cdot (x+3)/x$

13

Simplify: $(6x^2)/(9x)$

14

State restrictions: $5/(x^2 - 25)$

15

Simplify: $(x^2 - 4)/(x - 2)$. State restrictions.

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. What value is excluded from $x/(x - 5)$?

- **A.** $x = 0$
- **B.** $x = 5$
- **C.** $x = -5$
- **D.** None

2. What values are excluded from $3/(x^2 - 4)$?

- **A.** $x \neq 4$
- **B.** $x \neq \pm 2$
- **C.** $x \neq 2$
- **D.** $x \neq -4$

3. Simplify: $(3x^2)/(6x)$

- **A.** $x/2$
- **B.** $3/2$
- **C.** $x/6$
- **D.** $2x$

4. In rational expressions, you can cancel:

- **A.** Any matching numbers
- **B.** Common factors only
- **C.** Common terms only
- **D.** Any letters

5. Simplify: $(x^2 - 1)/(x - 1)$

- **A.** $x - 1$
- **B.** $x + 1$
- **C.** 1
- **D.** x^2

6. Simplify: $x/(x+3) \cdot (x+3)/x$

- **A.** 1
- **B.** x
- **C.** $x+3$
- **D.** 0

7. To divide $a/b \div c/d$:

- **A.** Multiply $a/b \cdot c/d$
- **B.** Multiply $a/b \cdot d/c$
- **C.** Subtract

  - **D.** Add

8. Simplify: (4x + 8)/4

  - **A.** x + 2
  - **B.** x + 8
  - **C.** 4x + 2
  - **D.** x + 4

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can identify the restrictions (excluded values) of a rational expression.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can factor numerators and denominators to simplify rational expressions.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can distinguish between cancelling common factors and common terms.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can simplify rational expressions to lowest terms.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can state all restrictions in the final answer.
Confident ☐ Mostly ☐ Need Review ☐

Unit 4 · Week 13

Multiplying & Dividing Rational Expressions

Multiply and divide rational expressions — applying the same rules you know for ordinary fractions, but with polynomial pieces.

LESSON OVERVIEW

Welcome to Week 13. We now extend our work with rational expressions to multiplication and division.

The rule for multiplying rational expressions is the same as for numerical fractions: multiply the numerators together and the denominators together.

In symbols: a over b times c over d equals a c over b d.

But before we multiply, we factor everything and cancel any common factors. This makes the final answer much simpler.

For division, we use the reciprocal trick. To divide by a fraction, flip it and multiply.

So a over b divided by c over d equals a over b times d over c.

After flipping, factor everything, then cancel common factors before multiplying.

Throughout the entire process, keep track of restrictions. Any value that ever made any denominator zero — at any stage — must be excluded.

If no common factors exist after factoring, simply multiply across. The result is your answer.

These skills are essential for the next lesson, where we will solve rational equations.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

$(a/b)\cdot(c/d) = ac/bd \cdot (a/b)\div(c/d) = (a/b)\cdot(d/c)$

💡 **DID YOU KNOW?**

The reciprocal rule for dividing fractions was first written down in ancient Egypt around 1650 BCE, in the Rhind Mathematical Papyrus — a document *used to train scribes in calculation.*

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1	**Multiply $x/(x+3) \cdot (x+3)/(x-2)$**
Step 1	Identify common factor: $(x+3)$
Step 2	Cancel
Step 3	Multiply remaining: $x/(x-2)$
Step 4	Restrictions: $x \neq -3, 2$
ANSWER	**$x/(x - 2), x \neq -3, 2$**

EXAMPLE 2	**Multiply $(x^2-9)/(x^2-x-6) \cdot (x-2)/(x+3)$**
Step 1	Factor: $(x-3)(x+3)/(x-3)(x+2) \cdot (x-2)/(x+3)$
Step 2	Cancel $(x-3)$ and $(x+3)$
Step 3	Multiply remaining
ANSWER	**$(x - 2)/(x + 2)$**

EXAMPLE 3	**Divide $x/(x-1) \div (x+2)/x$**
Step 1	Flip second fraction: $\cdot x/(x+2)$
Step 2	Multiply: $x^2/((x-1)(x+2))$
Step 3	State all restrictions
ANSWER	**$x^2/((x-1)(x+2)), x \neq 1, -2, 0$**

EXAMPLE 4 Divide $(x^2-4)/(x^2-9) \div (x-2)/(x+3)$

Step 1	Flip second: $\cdot(x+3)/(x-2)$
Step 2	Factor everything: $(x-2)(x+2)/(x-3)(x+3) \cdot (x+3)/(x-2)$
Step 3	Cancel $(x-2)$ and $(x+3)$

ANSWER $(x + 2)/(x - 3)$

EXAMPLE 5 Multiply $(x+1)/(x-4) \cdot (x+3)/(x+2)$

Step 1	Factor everything (already factored)
Step 2	Look for common factors
Step 3	None — just multiply

ANSWER $(x+1)(x+3)/((x-4)(x+2))$

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 Multiply: $x/(x+1) \cdot (x+1)/(x-2)$

2 Multiply: $3/x \cdot x/3$

3 Multiply: $4/(x+2) \cdot (x+2)/8$

4 Divide: $1/x \div 1/y$

5 Divide: $x/(x-1) \div (x+2)/x$. State restrictions.

6

Multiply: $(x^2-1)/x \cdot x/(x+1)$

7

Multiply: $(x^2-9)/(x^2-x-6) \cdot (x-2)/(x+3)$

8

Divide: $(x^2-4)/(x^2-9) \div (x-2)/(x+3)$

9

Multiply: $(x+1)/(x-4) \cdot (x+3)/(x+2)$

10

Divide: $x/(x+2) \div (x-1)/x$

11

Multiply: $2x/(x-1) \cdot (x-1)/(x+3)$

12

Divide: $5/x \div 5/y$

13

Multiply: $x^2/(x+5) \cdot (x+5)/x$

14

Divide: $(x^2+x)/(x-2) \div x/(x-2)$

15

Simplify: $(x/(x-4)) \cdot ((x-4)/(x+1))$

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. Multiply $x/(x+1) \cdot (x+1)/(x-2)$

- **A.** $x/(x-2)$
- **B.** $x(x+1)/(x-2)$
- **C.** $x/((x+1)(x-2))$
- **D.** 1

2. Divide $a/b \div c/d$ gives:

- **A.** $(a \cdot c)/(b \cdot d)$
- **B.** $(a \cdot d)/(b \cdot c)$
- **C.** $(b \cdot c)/(a \cdot d)$
- **D.** a/d

3. Multiply $(x^2-1)/x \cdot x/(x+1)$

- **A.** $x-1$
- **B.** $x+1$
- **C.** $(x^2-1)/(x+1)$
- **D.** x^2-1

4. Divide $1/x \div 1/y =$

- **A.** $1/(xy)$
- **B.** xy
- **C.** y/x
- **D.** x/y

5. Multiply $3/x \cdot x/3$

- **A.** 1
- **B.** 3
- **C.** x
- **D.** $x^2/9$

6. After multiplying fractions, you should:

- **A.** Add the answers
- **B.** Simplify by cancelling
- **C.** Subtract
- **D.** Square the answer

7. When the second fraction in a division has no common factors after flipping, you:

- **A.** Cannot solve
- **B.** Just multiply across

- **C.** Give up
- **D.** Add instead

8. Restrictions for x/(x−2) ÷ (x+1)/x include:

- **A.** $x \neq 2$ only
- **B.** $x \neq -1$ only
- **C.** $x \neq 2, -1, 0$
- **D.** $x \neq 0$

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can multiply two rational expressions and simplify.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can divide rational expressions using the reciprocal method.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can factor expressions before multiplying or dividing.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can identify and state all restrictions throughout the process.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can simplify rational expressions completely.
Confident ☐ Mostly ☐ Need Review ☐

Unit 4 · Week 14

Solving Rational Equations

Solve equations that contain fractions — using the Least Common Denominator to clear denominators and reveal the solution.

LESSON OVERVIEW

Welcome to Week 14. Today we solve rational equations — equations that contain rational expressions.

The strategy is simple but powerful: clear the denominators by multiplying every term by the Least Common Denominator, or LCD.

Step one: find the restrictions. Any value that makes any denominator zero must be excluded from the final answer.

Step two: find the LCD — the smallest expression that all denominators divide into.

Step three: multiply every term on both sides by the LCD. This eliminates the denominators.

Step four: solve the resulting equation. It will be linear or quadratic, depending on the original.

Step five — critically important — check every solution in the original equation. Some "solutions" might violate a restriction. These are called extraneous solutions and must be rejected.

For example, if you solve and get x equals 1, but x equals 1 makes a denominator zero, then x equals 1 is extraneous. It is not a true solution.

Sometimes the equation simplifies to a false statement like 0 equals negative 1. That means the equation has no solution.

By the end of today, you will solve rational equations and properly handle extraneous solutions.

KEY CONCEPTS & DID YOU KNOW

★ KEY FORMULA

1. Find restrictions 2. Multiply by LCD 3. Solve 4. Check for extraneous

💡 **DID YOU KNOW?**

The concept of an "extraneous solution" — a mathematical solution that turns out to violate the original problem — sounds strange but happens in many areas of mathematics, particularly with radicals and rational equations.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1 **Solve $x/(x-2) = 3$**	
Step 1	Restriction: $x \neq 2$
Step 2	Multiply both sides by $(x-2)$
Step 3	$x = 3(x - 2)$
Step 4	Expand: $x = 3x - 6$
Step 5	Solve: $-2x = -6$, $x = 3$
Step 6	Check: not excluded ✓
ANSWER	**$x = 3$**

EXAMPLE 2 **Solve $1/x + 2/(x+1) = 3$**	
Step 1	Restrictions: $x \neq 0, -1$
Step 2	LCD = $x(x+1)$
Step 3	Multiply: $(x+1) + 2x = 3x(x+1)$
Step 4	Simplify: $3x + 1 = 3x^2 + 3x$
Step 5	Solve: $3x^2 = 1$, $x^2 = 1/3$
Step 6	$x = \pm\sqrt{(1/3)}$
ANSWER	**$x = \pm\sqrt{(1/3)}$**

EXAMPLE 3 **Solve $x/(x-1) = 2/(x-1)$**

Step 1	Restriction: $x \neq 1$
Step 2	Multiply by $(x-1)$
Step 3	$x = 2$
Step 4	Check: $2 \neq 1$ ✓
ANSWER	**$x = 2$**

EXAMPLE 4	**Solve $x/(x-3) = 2/(x-3) + 1$**
Step 1	Restriction: $x \neq 3$
Step 2	Multiply by $(x-3)$: $x = 2 + (x-3)$
Step 3	Simplify: $x = x - 1$
Step 4	False statement: $0 = -1$
ANSWER	**No solution**

EXAMPLE 5	**Solve $1/x + 1/(x-2) = 1$**
Step 1	Restrictions: $x \neq 0, 2$
Step 2	LCD = $x(x-2)$
Step 3	Multiply: $(x-2) + x = x(x-2)$
Step 4	Simplify: $2x - 2 = x^2 - 2x$
Step 5	Rearrange: $x^2 - 4x + 2 = 0$
Step 6	Quadratic formula: $x = 2 \pm \sqrt{2}$
ANSWER	**$x = 2 \pm \sqrt{2}$**

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1

Solve: $x/(x - 5) = 2$

2

Solve: $x/(x - 2) = 3$

3

Solve: $1/x + 1/(x+2) = 3$

4

Solve: $1/x = 1/3$

5

Solve: $2/x = 1/4$

6

Solve: $x/(x-1) = (3/(x-1))$

7

Solve: $1/(x-2) - 1/x = 1$

8

Solve: $x/(x+1) + 2/(x-1) = 3$

9 Solve: $1/x + 1/(x-2) = 1$

10 Solve: $4/(x+1) = 2$

11 Solve: $5/x = 5/(x+2)$. Does this have a solution?

12 Solve: $x/(x+3) = 4$

13 Find the LCD of $1/x$ and $1/(x+2)$.

14 Find the LCD of $1/(x-1)$ and $1/(x+1)$.

15

Solve: $2/(x-3) = 4$

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. Solve: $x/(x-5) = 2$
- **A.** $x = 5$
- **B.** $x = 10$
- **C.** $x = 2$
- **D.** No solution

2. First step in solving rational equations:
- **A.** Multiply by LCD
- **B.** Identify restrictions
- **C.** Square both sides
- **D.** Add denominators

3. A solution that violates a restriction is called:
- **A.** Valid
- **B.** Approximate
- **C.** Extraneous
- **D.** Imaginary

4. Solve: $1/x = 1/3$
- **A.** $x = 1/3$
- **B.** $x = 3$
- **C.** $x = -3$
- **D.** $x = 0$

5. The LCD of $1/x$ and $1/(x+1)$ is:
- **A.** x
- **B.** $x+1$
- **C.** $x(x+1)$
- **D.** $2x+1$

6. Solve: $2/(x-1) = 4$
- **A.** $x = 1$
- **B.** $x = 1.5$
- **C.** $x = 2$
- **D.** $x = 0.5$

7. If solving gives $0 = 5$, the equation has:
- **A.** Infinite solutions
- **B.** One solution
- **C.** No solution

- **D.** Two solutions

8. After finding a solution, you must always:

- **A.** Square it
- **B.** Check it in the original equation
- **C.** Divide by 2
- **D.** Take its reciprocal

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can identify restrictions before solving a rational equation.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can find the least common denominator (LCD).
Confident ☐ Mostly ☐ Need Review ☐

✓ I can clear denominators by multiplying both sides by the LCD.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can detect and reject extraneous solutions.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can recognize when a rational equation has no solution.
Confident ☐ Mostly ☐ Need Review ☐

Unit 4 · Week 15

Applications of Rational Equations

Apply rational equations to real life — work problems, travel rates, and mixture problems, just like the merchants of Aksum once calculated shared work and travel.

LESSON OVERVIEW

Welcome to Week 15. This week, we apply rational equations to real-world situations.

The most common application is the work rate problem. If one worker takes 6 hours and another takes 3 hours, how long do they take working together?

The combined work formula says: 1 over T equals 1 over t one plus 1 over t two, where T is the total time and t one, t two are the individual times.

For our example: 1 over T equals 1 over 6 plus 1 over 3 equals 1 over 6 plus 2 over 6 equals 3 over 6 equals 1 over 2. So T equals 2 hours.

Another classic application is distance, rate, and time. The formula d equals r times t can be rearranged to t equals d over r — a rational expression.

If you travel 60 miles at x miles per hour and return at 40 miles per hour, with total time of 3 hours, you set up: 60 over x plus 60 over 40 equals 3, then solve for x.

Mixture problems involve combining substances of different concentrations. For example, mixing a 20 percent salt solution with a 40 percent solution to make a 30 percent solution.

In mixture problems, the amount of pure substance is constant: amount times percentage of one plus amount times percentage of the other equals total amount times target percentage.

In every application, define your variable clearly, set up the equation carefully, solve, then check that the answer makes physical sense — negative work times or impossible quantities should be rejected.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

Work: $1/T = 1/t_1 + 1/t_2$ · **Travel:** $t = d/r$

💡 **DID YOU KNOW?**

The combined work formula has been used since ancient times — Roman engineers used similar reasoning to plan how many workers were needed to build the famous aqueducts.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1 Two workers: 6 hrs and 3 hrs. Together?

Step 1	1/T = 1/6 + 1/3
Step 2	Common denominator 6: 1/T = 1/6 + 2/6
Step 3	1/T = 3/6 = 1/2
Step 4	T = 2 hours

ANSWER 2 hours

EXAMPLE 2 Travel: 60 mi at x mph, return at 40 mph, total 3 hrs

Step 1	60/x + 60/40 = 3
Step 2	60/x + 1.5 = 3
Step 3	60/x = 1.5
Step 4	x = 40 mph

ANSWER x = 40 mph

EXAMPLE 3 Mix 20% and 40% to get 10 L of 30%. How much 20%?

Step 1	Let x = liters of 20% solution
Step 2	0.2x + 0.4(10−x) = 0.3(10)
Step 3	0.2x + 4 − 0.4x = 3
Step 4	−0.2x = −1, x = 5

ANSWER **5 liters of 20% solution**

EXAMPLE 4 **Pipes: 4 hrs and 12 hrs. Together?**

Step 1	1/T = 1/4 + 1/12
Step 2	LCD 12: 3/12 + 1/12 = 4/12
Step 3	1/T = 1/3
Step 4	T = 3 hours

ANSWER **3 hours**

EXAMPLE 5 **Travel: at 30 mph, return at 60 mph, total 3 hrs. Distance?**

Step 1	d/30 + d/60 = 3
Step 2	Multiply by 60: 2d + d = 180
Step 3	3d = 180
Step 4	d = 60

ANSWER **60 miles**

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 Two workers take 5 and 10 hours individually. How long together?

2 Two workers take 4 and 12 hours. How long together?

3 Two pipes: 6 and 12 hours. How long to fill together?

4 A car travels 120 miles at x mph and returns at 60 mph in 5 hours. Find x.

5 A car travels 60 miles at x mph and returns at 40 mph in 3 hours. Find x.

6

Mix 30% and 50% solutions to get 20 liters of 40%. How much of each?

7

Mix 20% and 40% to get 10 liters of 30%. How much of each?

8

A person travels at 40 mph and returns at 80 mph in 4 hours. Find the one-way distance.

9 Two pipes: 8 hours and 6 hours. How long together?

10 A train travels d miles at 50 mph. Write time as function of d.

11 Two workers take 3 and 6 hours. How long together?

12 A car goes 80 mph for d miles. Express t in terms of d.

13 Mix 10% and 30% to get 15 liters of 20%. How much of each?

14 Three workers take 6, 4, and 12 hours. How long together?

15

Two cars start 200 mi apart and approach each other at 50 and 30 mph. When do they meet?

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. Two workers take 5 and 10 hours individually. Together (in hrs)?

- **A.** 7.5
- **B.** 3.33...
- **C.** 5
- **D.** 15

2. The work-together formula is 1/T = ___

- **A.** $t_1 + t_2$
- **B.** $1/t_1 + 1/t_2$
- **C.** $t_1 \times t_2$
- **D.** $t_1 - t_2$

3. Travel time formula: t = ___

- **A.** d × r
- **B.** d − r
- **C.** d/r
- **D.** r/d

4. Pipes: 8 hrs and 4 hrs. Together?

- **A.** 12 hrs
- **B.** 6 hrs
- **C.** 8/3 hrs
- **D.** 2 hrs

5. In mixture problems, what stays constant?

- **A.** Volume only
- **B.** Concentration
- **C.** Amount of pure substance
- **D.** Temperature

6. A car goes d miles at 40 mph, time =

- **A.** 40d
- **B.** d/40
- **C.** 40 − d
- **D.** 40 + d

7. Three workers take 2, 4, and 6 hours. Together (LCD = 12)?

- **A.** 1/T = 6+3+2 / 12
- **B.** 1/T = 1/2+1/4+1/6
- **C.** Both A and B

- **D.** Neither

8. After solving, you must check answers for:
- **A.** Algebra only
- **B.** Physical reasonableness
- **C.** Extraneous solutions
- **D.** Both B and C

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can set up and solve work rate problems with $1/T = 1/t_1 + 1/t_2$.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can solve distance-rate-time problems using rational equations.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can model and solve mixture problems.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can interpret solutions for physical reasonableness.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can apply rational equations across multiple real-world contexts.
Confident ☐ Mostly ☐ Need Review ☐

Unit 4 Answer Key

Compare your answers below. If you missed a question, review the corresponding lesson section.

Week 12: Simplifying Rational Expressions

Q#	Answer	Explanation
1	**B**	Set denominator ≠ 0: $x - 5 \neq 0$, so $x \neq 5$.
2	**B**	$x^2-4=(x-2)(x+2)$, so $x \neq 2$ and $x \neq -2$.
3	**A**	$3x^2/6x = (3/6)\cdot x = x/2$, $x \neq 0$.
4	**B**	Only common FACTORS (which multiply the whole expression) can be cancelled.
5	**B**	$(x-1)(x+1)/(x-1) = x+1$, $x \neq 1$.
6	**A**	Both common factors cancel: result is 1, $x \neq 0, -3$.
7	**B**	Dividing by a fraction means multiplying by its reciprocal.
8	**A**	Factor numerator: $4(x+2)/4 = x+2$.

Week 13: Multiplying & Dividing Rational Expressions

Q#	Answer	Explanation
1	**A**	$(x+1)$ cancels: $x/(x-2)$.
2	**B**	Flip and multiply: $a/b \cdot d/c = ad/bc$.
3	**A**	Factor: $(x-1)(x+1)/x \cdot x/(x+1) \rightarrow x-1$.
4	**C**	Flip: $1/x \cdot y/1 = y/x$.
5	**A**	Both cancel: 1, $x \neq 0$.
6	**B**	Always cancel common factors and simplify.
7	**B**	If no common factors, simply multiply numerators and denominators.
8	**C**	All denominators (original and reciprocal) contribute restrictions.

Week 14: Solving Rational Equations

Q#	Answer	Explanation
1	B	Multiply by (x−5): x = 2(x−5) = 2x−10, so x=10.
2	B	Always identify restrictions first to know which solutions to reject.
3	C	Extraneous solutions look correct algebraically but violate the original domain.
4	B	Cross-multiply: 3 = x, so x = 3.
5	C	The least common denominator is the product of the two distinct denominators.
6	B	Multiply by (x−1): 2 = 4(x−1) = 4x−4, so 4x=6, x=1.5.
7	C	A false statement (0=5) means no value satisfies the equation.
8	B	Check to ensure no restrictions are violated (no denominator becomes 0).

Week 15: Applications of Rational Equations

Q#	Answer	Explanation
1	B	1/T = 1/5 + 1/10 = 2/10 + 1/10 = 3/10, so T = 10/3 ≈ 3.33.
2	B	Combined rates add: $1/T = 1/t_1 + 1/t_2$.
3	C	From d = rt: t = d/r.
4	C	1/T = 1/8 + 1/4 = 3/8, so T = 8/3 hours.
5	C	The total amount of the substance is conserved.
6	B	t = d/r = d/40 hours.
7	C	Both express the same idea: combine rates with LCD.
8	D	Negative time, impossible amounts, and extraneous solutions must all be rejected.

UNIT 5

Radical Expressions & Equations

WEEKS COVERED

Week 16: Simplifying Radicals
Week 17: Operations with Radicals
Week 18: Solving Radical Equations

Unit 5 · Week 16

Simplifying Radicals

Tame the square root symbol — break radicals down to their simplest form, like a coffee roaster reducing beans to their purest essence.

LESSON OVERVIEW

Welcome to Unit 5. This week we begin working with radicals — expressions containing the square root symbol.

A radical is any expression that uses the root sign — most commonly the square root. The square root of 36 is 6, because 6 times 6 equals 36.

Many numbers are not perfect squares, but we can still simplify their roots by factoring out any perfect-square parts.

For example, the square root of 72. We notice 72 equals 36 times 2. The square root of 36 is 6. So the square root of 72 simplifies to 6 times the square root of 2.

This uses the product property of radicals: the square root of a times b equals the square root of a times the square root of b.

There is also a quotient property: the square root of a over b equals the square root of a divided by the square root of b.

When we have variables under a radical, we look for pairs. The square root of x squared is x. The square root of x squared times y is x times the square root of y.

A radical expression is in simplest form when no perfect-square factors remain inside the radical, and there are no radicals in the denominator.

When a radical appears in the denominator, we rationalize by multiplying the top and bottom by that same radical. For example, 1 over the square root of 5 becomes the square root of 5 over 5.

By the end of today, you will simplify radicals, apply product and quotient rules, and rationalize denominators.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \cdot \sqrt{a/b} = \sqrt{a} / \sqrt{b}$$

💡 **DID YOU KNOW?**

The radical symbol √ was invented in 1525 by German mathematician Christoff Rudolff. He chose it because it resembles the letter "r" — the first letter of the Latin word "radix," meaning "root."

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1 **Simplify $\sqrt{36}$**	
Step 1	Identify perfect square: $36 = 6^2$
Step 2	Apply: $\sqrt{(6^2)} = 6$
ANSWER **6**	

EXAMPLE 2 **Simplify $\sqrt{72}$**	
Step 1	Factor: $72 = 36 \times 2$
Step 2	Apply product rule: $\sqrt{72} = \sqrt{36} \cdot \sqrt{2}$
Step 3	Simplify: $6\sqrt{2}$
ANSWER **$6\sqrt{2}$**	

EXAMPLE 3 **Simplify $\sqrt{(x^2y)}$**	
Step 1	Separate: $\sqrt{(x^2)} \cdot \sqrt{y}$
Step 2	x^2 has perfect square: x
Step 3	y remains under root
ANSWER **$x\sqrt{y}$**	

EXAMPLE 4 **Rationalize $1/\sqrt{5}$**	
Step 1	Multiply top and bottom by $\sqrt{5}$

Step 2	(1·√5)/(√5·√5)
Step 3	Simplify denominator: √5 · √5 = 5
ANSWER	**√5/5**

EXAMPLE 5 **Simplify √18 / √2**

Step 1	Use quotient rule: √(18/2)
Step 2	Divide: √9
Step 3	Take root: 3
ANSWER	**3**

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 Simplify: $\sqrt{50}$

2 Simplify: $\sqrt{72}$

3 Simplify: $\sqrt{98}$

4 Simplify: $\sqrt{45}$

5 Simplify: $\sqrt{(x^2y^3)}$

6

Simplify: $\sqrt{16x^2}$

7

Simplify: $\sqrt{25y^4}$

8

Rationalize: $1/\sqrt{7}$

9

Rationalize: $1/\sqrt{5}$

10

Rationalize: $3/\sqrt{2}$

11

Simplify: $\sqrt{45}/\sqrt{5}$

12

Simplify: $\sqrt{48}/\sqrt{3}$

13

Simplify: $\sqrt{(36x^4)}$

14

Simplify: $\sqrt{200}$

15

Simplify: $\sqrt{(9x^2y)}$

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. Simplify $\sqrt{49}$

- **A.** 7
- **B.** 14
- **C.** 24.5
- **D.** 49

2. Simplify $\sqrt{50}$

- **A.** $5\sqrt{2}$
- **B.** $2\sqrt{5}$
- **C.** $25\sqrt{2}$
- **D.** $\sqrt{50}$

3. Simplify $\sqrt{98}$

- **A.** $7\sqrt{2}$
- **B.** $2\sqrt{7}$
- **C.** $49\sqrt{2}$
- **D.** 98

4. Rationalize $1/\sqrt{7}$

- **A.** $\sqrt{7}$
- **B.** 7
- **C.** $\sqrt{7}/7$
- **D.** $1/7$

5. Simplify $\sqrt{45}/\sqrt{5}$

- **A.** 3
- **B.** 9
- **C.** $\sqrt{9}$
- **D.** $\sqrt{45}/5$

6. Simplify $\sqrt{(16x^2)}$

- **A.** $4x$
- **B.** $4x^2$
- **C.** $16x$
- **D.** $\sqrt{16x}$

7. Product property: $\sqrt{a} \cdot \sqrt{b} =$

- **A.** $\sqrt{(a+b)}$
- **B.** $\sqrt{(ab)}$
- **C.** $\sqrt{a} + \sqrt{b}$

   D. $a \cdot b$

8. Simplify $\sqrt{x^2y^3}$

   A. $xy\sqrt{y}$
   B. x^2y
   C. $xy^2\sqrt{y}$
   D. $x\sqrt{y^3}$

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can simplify radicals by factoring out perfect squares.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can apply the product and quotient properties of radicals.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can simplify radicals with variables.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can rationalize denominators that contain a radical.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can express radicals in simplest form.
Confident ☐ Mostly ☐ Need Review ☐

Unit 5 · Week 17

Operations with Radicals

Add, subtract, multiply, and divide radicals — treating like radicals as algebra variables, distinct radicals as separate species.

LESSON OVERVIEW

Welcome to Week 17. This week we extend radical work to all four operations.

The most important concept: like radicals can be combined just like like terms in algebra.

Like radicals have the same value under the root. For example, 3 root 5 and 2 root 5 are like radicals. 3 root 5 and 2 root 7 are not.

To add or subtract like radicals, we combine the coefficients. For example, 3 root 5 plus 2 root 5 equals 5 root 5.

Unlike radicals cannot be combined directly. But sometimes we can simplify first, and then we discover they ARE alike.

For example, root 18 plus root 8. They look different — but root 18 simplifies to 3 root 2, and root 8 simplifies to 2 root 2. Now they are like radicals: 3 root 2 plus 2 root 2 equals 5 root 2.

For multiplication, we multiply the coefficients together and the radicands together. The product rule says root a times root b equals root a b.

For example, 2 root 3 times 4 root 6 equals 8 times root 18, which simplifies to 8 times 3 root 2, equals 24 root 2.

For division, we use the quotient rule. Root 50 divided by root 2 equals root 25, which equals 5.

Always simplify your final answer. And if a radical appears in the denominator, rationalize it.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

$$a\sqrt{c} + b\sqrt{c} = (a+b)\sqrt{c} \cdot \sqrt{a} \cdot \sqrt{b} = \sqrt{(ab)}$$

💡 **DID YOU KNOW?**

The ancient Egyptians, around 1800 BCE, were already calculating with square roots to design the precise dimensions of pyramids and temples — though they had no symbol for them.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1 Add: $3\sqrt{5} + 2\sqrt{5}$

Step 1	Both have $\sqrt{5}$ — like radicals
Step 2	Combine coefficients: $3 + 2 = 5$

ANSWER $5\sqrt{5}$

EXAMPLE 2 Subtract: $7\sqrt{3} - 4\sqrt{3}$

Step 1	Both have $\sqrt{3}$ — like radicals
Step 2	Combine: $7 - 4 = 3$

ANSWER $3\sqrt{3}$

EXAMPLE 3 Multiply: $\sqrt{2} \times \sqrt{18}$

Step 1	Use product rule: $\sqrt{(2 \cdot 18)}$
Step 2	$= \sqrt{36}$
Step 3	Take root

ANSWER 6

EXAMPLE 4 Multiply: $2\sqrt{3} \times 4\sqrt{6}$

Step 1	Coefficients: $2 \cdot 4 = 8$
Step 2	Radicands: $\sqrt{3} \cdot \sqrt{6} = \sqrt{18}$

Step 3	Simplify $\sqrt{18} = 3\sqrt{2}$
Step 4	Total: $8 \cdot 3\sqrt{2}$
ANSWER	$24\sqrt{2}$

EXAMPLE 5 Divide: $\sqrt{50} / \sqrt{2}$

Step 1	Use quotient rule: $\sqrt{(50/2)}$
Step 2	$= \sqrt{25}$
Step 3	Take root
ANSWER	5

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 Simplify: $5\sqrt{2} + 3\sqrt{2}$

2 Simplify: $6\sqrt{7} - 2\sqrt{7}$

3 Simplify: $\sqrt{3} + 2\sqrt{3} + 5\sqrt{3}$

4 Simplify: $4\sqrt{5} - \sqrt{5} + 2\sqrt{5}$

5 Multiply: $\sqrt{3} \times \sqrt{12}$

6

Multiply: $\sqrt{2} \times \sqrt{8}$

7

Multiply: $4\sqrt{5} \times 2\sqrt{10}$

8

Multiply: $3\sqrt{2} \times 5\sqrt{6}$

9 Rationalize: $8/\sqrt{3}$

10 Rationalize: $6/\sqrt{2}$

11 Divide: $\sqrt{50}/\sqrt{2}$

12 Divide: $\sqrt{80}/\sqrt{5}$

13 Simplify: $\sqrt{8} + \sqrt{2}$ (simplify first)

14 Simplify: $\sqrt{18} + \sqrt{8}$

15

Multiply: $(2 + \sqrt{3})(2 - \sqrt{3})$

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. Simplify: $5\sqrt{2} + 3\sqrt{2}$

- **A.** $8\sqrt{2}$
- **B.** $15\sqrt{2}$
- **C.** $8\sqrt{4}$
- **D.** 8

2. Simplify: $6\sqrt{7} - 2\sqrt{7}$

- **A.** $4\sqrt{7}$
- **B.** $8\sqrt{7}$
- **C.** 4
- **D.** $12\sqrt{7}$

3. Multiply: $\sqrt{3} \times \sqrt{12}$

- **A.** $\sqrt{36} = 6$
- **B.** 15
- **C.** $\sqrt{15}$
- **D.** $\sqrt{3}$

4. Multiply: $4\sqrt{5} \times 2\sqrt{10}$

- **A.** $8\sqrt{50}$
- **B.** $40\sqrt{2}$
- **C.** $8\sqrt{15}$
- **D.** $40\sqrt{2}$ simplified

5. Rationalize $8/\sqrt{3}$

- **A.** $8\sqrt{3}/3$
- **B.** $8/3$
- **C.** $\sqrt{24}$
- **D.** $24/3$

6. $3\sqrt{7} + 2\sqrt{7}$ simplifies because:

- **A.** They are unlike
- **B.** They are like radicals
- **C.** You always add radicals
- **D.** Both have a 7

7. $\sqrt{8} + \sqrt{2}$ simplifies first to:

- **A.** $\sqrt{10}$
- **B.** $2\sqrt{2} + \sqrt{2}$
- **C.** $3\sqrt{2}$

- **D.** Cannot combine

8. The product rule says $\sqrt{a} \cdot \sqrt{b}$ =

- **A.** $\sqrt{(a+b)}$
- **B.** $\sqrt{(ab)}$
- **C.** $a + b$
- **D.** $a \cdot b$

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can add and subtract like radicals.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can multiply radicals using the product rule, with and without coefficients.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can divide radicals using the quotient rule.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can simplify radical expressions before performing operations.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can rationalize denominators that contain radicals.
Confident ☐ Mostly ☐ Need Review ☐

Unit 5 · Week 18

Solving Radical Equations

Solve equations with radicals — isolate, square, solve, and crucially: check for "false solutions" that algebra produces but the original problem rejects.

LESSON OVERVIEW

Welcome to Week 18. This week we solve equations that contain a variable inside a radical.

The strategy: isolate the radical on one side, then square both sides to eliminate it.

Step one — isolate the radical. Move everything else to the other side of the equation.

Step two — square both sides. Since the square root of x, squared, equals x, this removes the radical.

Step three — solve the resulting equation. It will be linear or quadratic.

Step four — and critically important — check every solution in the original equation. Squaring both sides can introduce false solutions called extraneous solutions.

For example, solving root x equals negative 3 by squaring gives x equals 9. But root 9 is positive 3, not negative 3. So x equals 9 is extraneous.

When the equation has a variable on both sides — like root (x plus 5) equals x minus 1 — squaring produces a quadratic. Solve it, then test each answer.

Some radical equations have one valid solution and one extraneous. Some have two valid. Some have none. Always verify.

By the end of today, you will solve radical equations using the isolate-square-solve-check method.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

`Isolate · Square · Solve · CHECK`

💡 **DID YOU KNOW?**

The Pythagorean theorem — the most famous use of square roots — was *known to Babylonian mathematicians a thousand years before Pythagoras was born. They used it to survey land along the Tigris and Euphrates.*

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1 **Solve $\sqrt{x} = 5$**	
Step 1	Square both sides: $(\sqrt{x})^2 = 5^2$
Step 2	$x = 25$
Step 3	Check: $\sqrt{25} = 5$ ✓
ANSWER	$\mathbf{x = 25}$

EXAMPLE 2 **Solve $\sqrt{(x + 3)} = 4$**	
Step 1	Square: $x + 3 = 16$
Step 2	Solve: $x = 13$
Step 3	Check: $\sqrt{16} = 4$ ✓
ANSWER	$\mathbf{x = 13}$

EXAMPLE 3 **Solve $\sqrt{(x - 1)} + 2 = 6$**	
Step 1	Isolate: $\sqrt{(x - 1)} = 4$
Step 2	Square: $x - 1 = 16$
Step 3	Solve: $x = 17$
Step 4	Check ✓
ANSWER	$\mathbf{x = 17}$

EXAMPLE 4 Solve $\sqrt{2x + 1} = x - 1$

Step 1	Square: $2x+1 = (x-1)^2$
Step 2	Expand: $2x+1 = x^2 - 2x + 1$
Step 3	Rearrange: $x^2 - 4x = 0$
Step 4	Factor: $x(x-4) = 0$
Step 5	Test $x=0$: $\sqrt{1}=-1$ ✗ (extraneous)
Step 6	Test $x=4$: $\sqrt{9}=3$ ✓

ANSWER $x = 4$

EXAMPLE 5 Solve $\sqrt{x + 5} = x - 1$

Step 1	Square: $x+5 = (x-1)^2$
Step 2	Expand: $x+5 = x^2 - 2x + 1$
Step 3	Rearrange: $x^2 - 3x - 4 = 0$
Step 4	Factor: $(x-4)(x+1) = 0$
Step 5	$x=4$: $\sqrt{9}=3$ ✓
Step 6	$x=-1$: $\sqrt{4}=-2$ ✗ (extraneous)

ANSWER $x = 4$

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 Solve: $\sqrt{x} = 7$

2 Solve: $\sqrt{x} = 9$

3 Solve: $\sqrt{(x + 9)} = 5$

4 Solve: $\sqrt{(x - 4)} = 6$

5 Solve: $\sqrt{(2x + 3)} = 3$

6

Solve: $\sqrt{(3x - 2)} = 4$

7

Solve: $\sqrt{(x - 4)} + 1 = 6$

8

Solve: $\sqrt{(x + 5)} - 2 = 3$

9

Solve: $\sqrt{x + 2} = x - 2$ (check for extraneous)

10

Solve: $\sqrt{2x + 1} = x - 1$ (check for extraneous)

11

Solve: $\sqrt{x} = -4$ (any solution?)

12

Solve: $\sqrt{x + 7} = 4$

13

Solve: $\sqrt{5x - 1} = 7$

14

Solve: $\sqrt{x^2 - 9} = 4$

15

Solve: $\sqrt{(3x + 1)} - 2 = 0$

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. Solve: $\sqrt{x} = 7$

- **A.** $x = 7$
- **B.** $x = 49$
- **C.** $x = \sqrt{7}$
- **D.** $x = 14$

2. Solve: $\sqrt{(x + 9)} = 5$

- **A.** $x = 4$
- **B.** $x = 16$
- **C.** $x = 25$
- **D.** $x = -4$

3. Solve: $\sqrt{(2x + 3)} = 3$

- **A.** $x = 0$
- **B.** $x = 3$
- **C.** $x = 6$
- **D.** $x = 9$

4. Solve: $\sqrt{(x - 4)} + 1 = 6$

- **A.** $x = 25$
- **B.** $x = 29$
- **C.** $x = 27$
- **D.** $x = 21$

5. First step in solving radical equations:

- **A.** Add 1
- **B.** Isolate the radical
- **C.** Multiply by 2
- **D.** Square right away

6. After squaring both sides, you MUST:

- **A.** Take square root again
- **B.** Check for extraneous solutions
- **C.** Add 1 to both sides
- **D.** Stop

7. An extraneous solution:

- **A.** Is a real number
- **B.** Solves the original equation
- **C.** Looks correct algebraically but does NOT satisfy original

- **D.** Is always negative

8. The principal square root of 25 equals:

- **A.** 5
- **B.** −5
- **C.** Both +5 and −5
- **D.** 25

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can isolate a radical on one side of an equation.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can square both sides of an equation to eliminate a radical.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can solve the resulting equation (linear or quadratic).
Confident ☐ Mostly ☐ Need Review ☐

✓ I can detect and reject extraneous solutions.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can verify solutions in the original equation.
Confident ☐ Mostly ☐ Need Review ☐

Unit 5 Answer Key

Compare your answers below. If you missed a question, review the corresponding lesson section.

Week 16: Simplifying Radicals

Q#	Answer	Explanation
1	**A**	$49 = 7^2$, so $\sqrt{49} = 7$.
2	**A**	$50 = 25 \cdot 2$, so $\sqrt{50} = 5\sqrt{2}$.
3	**A**	$98 = 49 \cdot 2$, so $\sqrt{98} = 7\sqrt{2}$.
4	**C**	Multiply top & bottom by $\sqrt{7}$: $\sqrt{7}/(\sqrt{7}\cdot\sqrt{7}) = \sqrt{7}/7$.
5	**A**	Quotient rule: $\sqrt{(45/5)} = \sqrt{9} = 3$.
6	**A**	$\sqrt{16} = 4$, $\sqrt{(x^2)} = x$, so $4x$.
7	**B**	$\sqrt{a} \cdot \sqrt{b} = \sqrt{(ab)}$.
8	**A**	$\sqrt{(x^2)} \cdot \sqrt{(y^2\cdot y)} = x \cdot y\sqrt{y} = xy\sqrt{y}$.

Week 17: Operations with Radicals

Q#	Answer	Explanation
1	**A**	Like radicals — add coefficients: $5+3 = 8$.
2	**A**	Like radicals: $6 - 2 = 4$.
3	**A**	Product rule: $\sqrt{(3\cdot 12)} = \sqrt{36} = 6$.
4	**B**	Coefficients $4\cdot 2=8$, radicands $\sqrt{(5\cdot 10)}=\sqrt{50}=5\sqrt{2}$, total: $8\cdot 5\sqrt{2} = 40\sqrt{2}$.
5	**A**	Multiply by $\sqrt{3}/\sqrt{3}$: $8\sqrt{3}/3$.
6	**B**	Same radicand — like radicals can be added.
7	**B**	$\sqrt{8} = 2\sqrt{2}$, so $\sqrt{8} + \sqrt{2} = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2}$.
8	**B**	$\sqrt{a} \cdot \sqrt{b} = \sqrt{(ab)}$.

Week 18: Solving Radical Equations

Q#	Answer	Explanation
1	**B**	Square both sides: x = 49.
2	**B**	Square: x+9 = 25, so x = 16. Check: √16+9 ... wait √(16+9) = √25 = 5 ✓.
3	**B**	Square: 2x+3 = 9, so 2x = 6, x = 3.
4	**B**	Isolate: √(x−4)=5. Square: x−4=25, so x=29.
5	**B**	Always isolate the radical first.
6	**B**	Squaring can introduce extraneous solutions — always verify.
7	**C**	Extraneous solutions arise from squaring but fail in the original equation.
8	**A**	The radical symbol denotes the principal (positive) root: √25 = 5.

UNIT 6

Quadratic Functions & Equations

WEEKS COVERED

Week 19: Graphing Quadratics
Week 20: Solving Quadratics by Factoring
Week 21: Completing the Square
Week 22: The Quadratic Formula
Week 23: Applications of Quadratics

Unit 6 · Week 19

Graphing Quadratics

Meet the parabola — the elegant U-shaped curve traced by every thrown stone, every falling coffee bean, every arch in Lalibela.

LESSON OVERVIEW

Welcome to Unit 6. This week we begin the study of quadratic functions — perhaps the most beautiful curves in algebra.

A quadratic function has the form y equals a x squared plus b x plus c, where a is not zero.

Its graph is called a parabola — a graceful U-shaped curve. The same curve traced by a ball thrown into the air, or by water rising from a fountain.

The highest or lowest point of the parabola is called the vertex. The vertical line passing through the vertex is the axis of symmetry — it divides the parabola into two mirror-image halves.

The coefficient a determines the shape. If a is positive, the parabola opens upward — like a smile. If a is negative, it opens downward — like a frown.

Larger absolute values of a make the parabola narrower. Smaller absolute values make it wider.

To find the axis of symmetry, use the formula x equals negative b over 2 a. To find the y-coordinate of the vertex, substitute that x-value back into the original equation.

The y-intercept of a parabola is the value c — the point where x equals zero. The x-intercepts, where they exist, are found by setting y to zero and solving.

Parabolas describe projectile motion, optimal pricing, the curve of arches, and much more.

By the end of today, you will graph quadratic functions, find vertices, axes of symmetry, and intercepts.

KEY CONCEPTS & DID YOU KNOW

★ KEY FORMULA

Axis of symmetry: $x = -b/(2a)$ · Vertex: $(-b/2a, f(-b/2a))$

💡 **DID YOU KNOW?**

The famous rock-hewn churches of Lalibela, carved into the Ethiopian highlands in the 12th century, feature roof arches that closely follow parabolic curves — though their builders had no algebra to describe them.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1	**Find vertex of $y = x^2 - 4x + 3$**
Step 1	$a=1$, $b=-4$, $c=3$
Step 2	Axis: $x = -(-4)/2(1) = 2$
Step 3	$y = 4 - 8 + 3 = -1$
ANSWER	**Vertex (2, −1)**

EXAMPLE 2	**Find intercepts of $y = x^2 - 9$**
Step 1	y-intercept ($x=0$): $y = -9$
Step 2	x-intercepts ($y=0$): $x^2 = 9$
Step 3	$x = \pm 3$
ANSWER	**(−3,0), (3,0), (0,−9)**

EXAMPLE 3	**Graph $y = x^2$ (table of values)**
Step 1	$x=-2$: $y=4$ · $x=-1$: $y=1$ · $x=0$: $y=0$
Step 2	$x=1$: $y=1$ · $x=2$: $y=4$
Step 3	Vertex at origin
Step 4	U-shape opening upward
ANSWER	**Parabola, vertex (0,0)**

EXAMPLE 4 **Describe $y = -2x^2$**	
Step 1	$a = -2$ → opens downward
Step 2	$\|a\|=2$ → narrower than $y=x^2$
Step 3	$b=0$, $c=0$ → vertex at origin
ANSWER	**Downward, narrow, vertex (0,0)**

EXAMPLE 5 **Ball height: $y = -x^2 + 6x$. Max height?**	
Step 1	$a=-1$, $b=6$
Step 2	Axis: $x = -6/(-2) = 3$
Step 3	$y = -9 + 18 = 9$
ANSWER	**Maximum height 9 at $x = 3$**

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 Find the vertex of $y = x^2 + 6x + 5$

2 Find the vertex of $y = x^2 - 4x + 3$

3 Find the vertex of $y = 2x^2 - 8x + 1$

4 Find the intercepts of $y = x^2 - 16$

5 Find the intercepts of $y = x^2 - 25$

6

Describe the effect on $y = x^2$ when changed to $y = -x^2 + 4$

7

Describe the effect on $y = x^2$ when changed to $y = 3x^2$

8

Graph $y = x^2 - 2x$ using a table of values from $x = -1$ to $x = 4$.

9 Graph $y = x^2 + 2$ using a table of values.

10 Identify a, b, c in $y = 4x^2 + 7x - 2$.

11 Identify a, b, c in $y = -x^2 + 5x - 1$.

12 What does the sign of a tell you about the parabola?

13 Find the axis of symmetry of $y = x^2 + 8x + 12$.

14 Find y-intercept of $y = 3x^2 - 5x + 7$.

15

For $y = (x - 4)^2 + 2$, identify the vertex.

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. In $y = x^2 - 6x + 8$, the axis of symmetry is:
- **A.** $x = 6$
- **B.** $x = 3$
- **C.** $x = -3$
- **D.** $x = 8$

2. If $a > 0$, the parabola opens:
- **A.** Downward
- **B.** Upward
- **C.** Sideways
- **D.** Doesn't graph

3. For $y = -3x^2 + 1$, the parabola opens:
- **A.** Upward
- **B.** Downward
- **C.** Sideways
- **D.** Not a parabola

4. The vertex of $y = (x - 4)^2 + 2$ is:
- **A.** (4, 2)
- **B.** (−4, 2)
- **C.** (2, 4)
- **D.** (−4, −2)

5. y-intercept of $y = 2x^2 + 3x - 7$:
- **A.** 2
- **B.** 3
- **C.** −7
- **D.** 7

6. x-intercepts of $y = x^2 - 25$:
- **A.** $x = 25$
- **B.** $x = \pm 5$
- **C.** $x = 5$
- **D.** $x = \pm 25$

7. A parabola with $a = 4$ vs $a = 1$ is:
- **A.** Wider
- **B.** Narrower
- **C.** Same shape

D. Inverted

8. The axis of symmetry passes through:

A. The y-intercept
B. The vertex
C. The x-intercepts
D. Origin always

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can identify the vertex and axis of symmetry of a parabola.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can determine whether a parabola opens upward or downward from its equation.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can find the x- and y-intercepts of a quadratic function.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can graph quadratic functions using a table of values.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can interpret the effects of a, b, and c on a parabola.
Confident ☐ Mostly ☐ Need Review ☐

Unit 6 · Week 20

Solving Quadratics by Factoring

Use the Zero Product Property to solve quadratic equations — a centuries-old trick that turns one equation into two.

LESSON OVERVIEW

Welcome to Week 20. This week we learn the most efficient method for solving quadratic equations: factoring.

The technique uses one beautiful idea — the Zero Product Property. If two factors multiply to give zero, then at least one of them must be zero.

In symbols: if a times b equals zero, then either a equals zero or b equals zero, or both.

To solve a quadratic by factoring, follow four steps. First, move all terms to one side so the equation equals zero. Second, factor completely. Third, set each factor equal to zero. Fourth, solve each smaller equation.

For example, solve x squared plus five x plus six equals zero. Factor: x plus 2 times x plus 3 equals zero. Set each factor to zero: x equals negative 2 or x equals negative 3.

When the leading coefficient is not one, like 2 x squared plus 7 x plus 3, use the AC method or trial-and-error to factor.

When you see a difference of squares, like x squared minus 16, factor as (x minus 4)(x plus 4). Solutions: x equals plus or minus 4.

Always look for a GCF first. For 3 x squared plus 6 x, factor out 3 x: 3 x times (x plus 2) equals zero. Solutions: x equals 0 or x equals negative 2.

In real-world problems, a ball thrown into the air follows a quadratic path. Setting height to zero finds when the ball hits the ground.

By the end of today, you will solve any factorable quadratic equation using the Zero Product Property.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

$$\text{If } a\cdot b = 0, \text{ then } a = 0 \text{ or } b = 0$$

💡 **DID YOU KNOW?**

The Zero Product Property may seem simple, but it took mathematicians over 1,000 years to formalize it. Indian mathematician Brahmagupta wrote about it in 628 CE.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1	**Solve $x^2 + 5x + 6 = 0$**
Step 1	Factor: $(x+2)(x+3) = 0$
Step 2	Set each factor to zero
Step 3	$x = -2$ or $x = -3$
ANSWER	**$x = -2, -3$**

EXAMPLE 2	**Solve $2x^2 + 7x + 3 = 0$**
Step 1	Factor: $(2x+1)(x+3) = 0$
Step 2	$2x+1 = 0 \rightarrow x = -1/2$
Step 3	$x+3 = 0 \rightarrow x = -3$
ANSWER	**$x = -1/2, -3$**

EXAMPLE 3	**Solve $x^2 - 16 = 0$**
Step 1	Difference of squares: $(x-4)(x+4) = 0$
Step 2	$x = 4$ or $x = -4$
ANSWER	**$x = \pm 4$**

EXAMPLE 4	**Solve $3x^2 + 6x = 0$**
Step 1	Factor GCF: $3x(x + 2) = 0$

Step 2	$3x = 0 \rightarrow x = 0$
Step 3	$x+2 = 0 \rightarrow x = -2$
ANSWER	**$x = 0, -2$**

EXAMPLE 5	**When does $h = -x^2 + 4x$ hit the ground?**
Step 1	Set $h = 0$: $-x^2 + 4x = 0$
Step 2	Factor: $-x(x - 4) = 0$
Step 3	$x = 0$ (start) or $x = 4$ (lands)
ANSWER	**Ball hits ground at $x = 4$**

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 Solve: $x^2 + 7x + 10 = 0$

2 Solve: $x^2 + 5x + 6 = 0$

3 Solve: $x^2 - 9 = 0$

4 Solve: $x^2 - 25 = 0$

5 Solve: $2x^2 + 5x + 2 = 0$

6

Solve: $3x^2 + 12x = 0$

7

Solve: $4x^2 - 16 = 0$

8

Solve: $x^2 - 6x + 8 = 0$

9

Solve: $x^2 - 5x - 14 = 0$

10

Solve: $x^2 + 3x - 10 = 0$

11

A ball: $h = -x^2 + 4x$. When does it hit the ground?

12

Solve: $x^2 - 7x + 12 = 0$

13

Solve: $2x^2 - 8 = 0$

14

Solve: $x^2 + 9x + 14 = 0$

15

Solve: $x^2 - 11x + 30 = 0$

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. Solve $x^2 + 7x + 10 = 0$

- **A.** $x = 2, 5$
- **B.** $x = -2, -5$
- **C.** $x = -10, 7$
- **D.** $x = -7, -10$

2. Solve $x^2 - 9 = 0$

- **A.** $x = 9$
- **B.** $x = \pm 3$
- **C.** $x = \pm 9$
- **D.** $x = 3$

3. The Zero Product Property says:

- **A.** Zero times anything is zero
- **B.** If $ab=0$, then $a=0$ or $b=0$
- **C.** Products are zero
- **D.** $a \times b = a + b$

4. Solve $2x^2 + 5x + 2 = 0$

- **A.** $x = -2, -1/2$
- **B.** $x = 2, 1/2$
- **C.** $x = -2, 1/2$
- **D.** $x = 2, -1/2$

5. Solve $3x^2 + 12x = 0$

- **A.** $x = 0, 4$
- **B.** $x = 0, -4$
- **C.** $x = \pm 4$
- **D.** $x = 3, 12$

6. Solve $x^2 - 6x + 8 = 0$

- **A.** $x = 2, 4$
- **B.** $x = -2, -4$
- **C.** $x = 4, 6$
- **D.** $x = 1, 8$

7. First step in solving any quadratic by factoring:

- **A.** Take square root
- **B.** Set equation = 0
- **C.** Multiply by 2

- **D.** Add 1

8. For a height equation $h = -x^2 + 6x$, hits the ground when:

- **A.** $x = 0$ only
- **B.** $x = 6$ only
- **C.** $x = 0$ or 6
- **D.** $x = 3$

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can set a quadratic equation equal to zero.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can apply the Zero Product Property to solve quadratics.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can factor and solve quadratic equations.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can recognize and solve special cases (GCF, difference of squares).
Confident ☐ Mostly ☐ Need Review ☐

✓ I can solve real-world quadratic problems by factoring.
Confident ☐ Mostly ☐ Need Review ☐

Unit 6 · Week 21

Completing the Square

Transform any quadratic into a perfect square — a powerful method that reveals the vertex and solves what factoring cannot.

LESSON OVERVIEW

Welcome to Week 21. This week we learn the technique of completing the square — a beautiful method invented by Persian mathematician Al-Khwarizmi over a thousand years ago.

Some quadratics cannot be factored nicely. For these, we use a different approach: we transform the equation into a perfect square trinomial.

A perfect square trinomial is one that factors into a binomial squared. For example, x squared plus six x plus nine is a perfect square because it equals (x plus 3) squared.

To complete the square, follow these steps. First, make sure the coefficient of x squared is 1. Second, move the constant to the other side. Third, take half of the x coefficient, square it, and add it to BOTH sides. Fourth, factor as a perfect square. Fifth, solve.

For example, x squared plus six x plus five equals zero. Move 5: x squared plus six x equals negative 5. Half of 6 is 3, squared is 9. Add 9 to both sides: x squared plus six x plus 9 equals 4. Factor: (x plus 3) squared equals 4. Take square roots: x plus 3 equals plus or minus 2. Solve: x equals negative 1 or negative 5.

Completing the square is also how we convert a quadratic into vertex form: y equals a (x minus h) squared plus k. The vertex is at (h, k).

This form is extremely useful: it instantly reveals the vertex, the maximum or minimum value, and the axis of symmetry.

For example, y equals x squared plus four x plus one. Complete the square: y equals (x plus 2) squared minus 3. Vertex: (negative 2, negative 3).

By the end of today, you will complete the square to solve quadratic equations and convert to vertex form.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

Add $(b/2)^2$ to both sides to form a perfect square

💡 **DID YOU KNOW?**

The phrase "completing the square" comes from the geometric origin of the technique — al-Khwarizmi visualized $x^2 + bx$ as a literal square plus a rectangle, then asked: what piece would complete this into a perfect square?

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1 Solve $x^2 + 6x + 5 = 0$ by completing the square

Step 1	Move constant: $x^2 + 6x = -5$
Step 2	Half of 6 is 3; $3^2 = 9$
Step 3	Add 9 to both sides: $x^2 + 6x + 9 = 4$
Step 4	Factor: $(x + 3)^2 = 4$
Step 5	Square root: $x + 3 = \pm 2$
Step 6	Solve: $x = -1$ or -5

ANSWER $x = -1, -5$

EXAMPLE 2 Solve $x^2 - 8x + 3 = 0$

Step 1	Move: $x^2 - 8x = -3$
Step 2	Half of -8 is -4; $(-4)^2 = 16$
Step 3	Add 16: $x^2 - 8x + 16 = 13$
Step 4	Factor: $(x - 4)^2 = 13$
Step 5	$x - 4 = \pm\sqrt{13}$

ANSWER $x = 4 \pm \sqrt{13}$

EXAMPLE 3 Convert $y = x^2 + 4x + 1$ to vertex form

Step 1	Group x-terms: $y = (x^2 + 4x) + 1$

Step 2	Complete square: half of 4 is 2; $2^2 = 4$
Step 3	$y = (x^2 + 4x + 4) + 1 - 4$
Step 4	$y = (x + 2)^2 - 3$

ANSWER **Vertex: (−2, −3)**

EXAMPLE 4 **Find vertex of $y = (x - 5)^2 + 7$**

Step 1	Already in vertex form $y = a(x-h)^2 + k$
Step 2	$h = 5$, $k = 7$

ANSWER **Vertex: (5, 7)**

EXAMPLE 5 **Find max of $y = -x^2 + 6x$**

Step 1	Factor out −1: $y = -(x^2 - 6x)$
Step 2	Complete: half of −6 is −3; 9
Step 3	$y = -(x^2 - 6x + 9) + 9$
Step 4	$y = -(x - 3)^2 + 9$

ANSWER **Max value 9 at $x = 3$**

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1

Solve by completing the square: $x^2 + 10x + 9 = 0$

2

Solve by completing the square: $x^2 - 6x + 2 = 0$

3

Solve by completing the square: $x^2 + 8x + 7 = 0$

4

What number completes the square: $x^2 + 12x +$ ___?

5

What number completes the square: $x^2 - 14x +$ ___?

6

Convert to vertex form: $y = x^2 - 4x + 6$

7

Convert to vertex form: $y = x^2 + 6x + 5$

8

Find the vertex of $y = x^2 + 2x - 3$ (by completing the square)

9 Find the vertex of $y = (x - 5)^2 + 7$

10 Convert to vertex form and identify max/min: $y = -x^2 + 6x$

11 What is the vertex of $y = (x + 3)^2 - 4$?

12 Solve: $x^2 - 4x - 5 = 0$ by completing the square.

13 Solve: $x^2 + 2x = 8$

14 Convert: $y = x^2 + 10x + 21$

15 Find vertex by completing the square: $y = x^2 - 2x + 6$

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. To complete the square of $x^2 + 8x$, add:

- **A.** 4
- **B.** 8
- **C.** 16
- **D.** 64

2. To complete the square of $x^2 - 10x$, add:

- **A.** 5
- **B.** 25
- **C.** 50
- **D.** 100

3. Vertex form is:

- **A.** $y = ax^2 + bx + c$
- **B.** $y = a(x - h)^2 + k$
- **C.** $y = a/x$
- **D.** $y = ax + b$

4. In $y = 2(x - 3)^2 + 5$, vertex is:

- **A.** (3, 5)
- **B.** (−3, 5)
- **C.** (3, −5)
- **D.** (2, 5)

5. When completing the square, you add the same value to:

- **A.** Only left side
- **B.** Only right side
- **C.** Both sides
- **D.** Neither side

6. $x^2 + 6x + 9$ factors as:

- **A.** $(x + 9)^2$
- **B.** $(x + 3)^2$
- **C.** $(x + 6)^2$
- **D.** $(x - 3)^2$

7. Convert $y = x^2 + 4x$ to vertex form:

- **A.** $(x+2)^2 - 4$
- **B.** $(x+2)^2 + 4$
- **C.** $(x+4)^2 - 4$

- **D.** $(x-2)^2 - 4$

8. The vertex of $y = (x + 2)^2 - 4$ is:
- **A.** (2, 4)
- **B.** (−2, −4)
- **C.** (2, −4)
- **D.** (−2, 4)

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can complete the square to form a perfect square trinomial.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can solve quadratic equations using completing the square.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can convert from standard form to vertex form.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can identify the vertex from vertex form.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can apply completing the square to optimization problems.
Confident ☐ Mostly ☐ Need Review ☐

Unit 6 · Week 22

The Quadratic Formula

The universal solution — one formula that solves every quadratic equation, no matter how unfactorable.

LESSON OVERVIEW

Welcome to Week 22. This week we meet one of the most famous formulas in mathematics: the quadratic formula.

The quadratic formula solves any equation in the form a x squared plus b x plus c equals zero.

The formula is: x equals negative b plus or minus the square root of b squared minus 4 a c, all divided by 2 a.

To use it, write the equation in standard form, identify a, b, and c, then substitute carefully.

For example, x squared plus 5 x plus 6 equals zero. Here a equals 1, b equals 5, c equals 6.

Substitute: x equals negative 5 plus or minus the square root of 25 minus 24, all over 2. That simplifies to negative 5 plus or minus 1, over 2. So x equals negative 2 or negative 3.

The expression under the square root — b squared minus 4 a c — is called the discriminant. It tells us how many solutions to expect.

If the discriminant is positive, there are two real solutions. If it equals zero, there is one repeated real solution. If it is negative, there are no real solutions — only complex ones.

The quadratic formula is universal. Factoring may be faster when it works, but the formula always works, even for messy equations.

By the end of today, you will apply the quadratic formula to solve any quadratic equation.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

$x = (-b \pm \sqrt{b^2 - 4ac}) / (2a)$ · Discriminant: $b^2 - 4ac$

💡 **DID YOU KNOW?**

Although named after no one specific person, the quadratic formula in its modern form was perfected by mathematicians like Bhāskara II in India (1100s) and later European algebraists.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1 **Solve $x^2 + 5x + 6 = 0$**

Step 1	a=1, b=5, c=6
Step 2	Discriminant: $25 - 24 = 1$
Step 3	$x = (-5 \pm \sqrt{1})/2$
Step 4	$x = (-5 + 1)/2 = -2$, $x = (-5 - 1)/2 = -3$

ANSWER **$x = -2, -3$**

EXAMPLE 2 **Solve $2x^2 + 3x - 2 = 0$**

Step 1	a=2, b=3, c=−2
Step 2	Disc: $9 + 16 = 25$
Step 3	$x = (-3 \pm 5)/4$
Step 4	$x = 2/4 = 1/2$, $x = -8/4 = -2$

ANSWER **$x = 1/2, -2$**

EXAMPLE 3 **Solve $x^2 + 4x + 4 = 0$**

Step 1	a=1, b=4, c=4
Step 2	Disc: $16 - 16 = 0$
Step 3	One repeated solution
Step 4	$x = -4/2 = -2$

ANSWER **$x = -2$ (double root)**

EXAMPLE 4 **Solve $x^2 + x + 5 = 0$**

Step 1	a=1, b=1, c=5
Step 2	Disc: $1 - 20 = -19$
Step 3	Negative → no real solutions

ANSWER **No real solutions**

EXAMPLE 5 **Solve $x^2 - 4x + 1 = 0$**

Step 1	a=1, b=−4, c=1
Step 2	Disc: $16 - 4 = 12$
Step 3	$x = (4 \pm \sqrt{12})/2 = (4 \pm 2\sqrt{3})/2$
Step 4	$x = 2 \pm \sqrt{3}$

ANSWER **$x = 2 \pm \sqrt{3}$**

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1

Solve: $x^2 + 7x + 10 = 0$

2

Solve: $2x^2 + x - 3 = 0$

3

Solve: $x^2 - 4x + 1 = 0$

4

Solve: $x^2 + 2x + 10 = 0$ (find discriminant first)

5

Solve: $3x^2 - 6x + 2 = 0$

6 Compute the discriminant: $x^2 + 3x + 5 = 0$. Number of real solutions?

7 Compute the discriminant: $x^2 - 4x + 4 = 0$. Number of real solutions?

8 Compute the discriminant: $x^2 + 5x + 6 = 0$. Number of real solutions?

9 Solve: $x^2 + 8x + 16 = 0$

10 Solve: $2x^2 - 5x + 1 = 0$

11 Solve: $x^2 - 2x - 8 = 0$

12 Solve: $3x^2 + 7x + 2 = 0$

13 Identify a, b, c in $4x^2 - 3x + 7 = 0$.

14 Identify a, b, c in $x^2 + 9 = 6x$. (Hint: rearrange first)

15

Solve: $x^2 + 4x - 12 = 0$

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. The quadratic formula is:
- **A.** $x = b \pm \sqrt{(a^2 - 4bc)} / 2a$
- **B.** $x = (-b \pm \sqrt{(b^2 - 4ac)}) / 2a$
- **C.** $x = -b/2a$
- **D.** $x = (b^2 - 4ac) / 2a$

2. The discriminant is:
- **A.** $b^2 + 4ac$
- **B.** $b^2 - 4ac$
- **C.** $b - 4ac$
- **D.** $-b/2a$

3. If $b^2 - 4ac > 0$, the equation has:
- **A.** No real solutions
- **B.** One solution
- **C.** Two real solutions
- **D.** Complex solutions

4. If $b^2 - 4ac = 0$, the equation has:
- **A.** No solutions
- **B.** One repeated real solution
- **C.** Two real solutions
- **D.** Negative solutions

5. For $3x^2 - 6x + 2 = 0$: a, b, c are:
- **A.** 3, −6, 2
- **B.** 3, 6, 2
- **C.** −3, 6, −2
- **D.** 3, −6, −2

6. For $x^2 + 7x + 10 = 0$, solutions are:
- **A.** $x = -2, -5$
- **B.** $x = 2, 5$
- **C.** $x = -7, -10$
- **D.** $x = -1, -10$

7. For $x^2 + 2x + 10 = 0$, the equation has:
- **A.** Two real solutions
- **B.** One real solution
- **C.** No real solutions

- **D.** Infinite solutions

8. The quadratic formula works for:

- **A.** Only factorable quadratics
- **B.** Only when a = 1
- **C.** Any quadratic equation
- **D.** Only positive discriminants

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can identify the coefficients a, b, and c in a quadratic equation.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can apply the quadratic formula to solve any quadratic equation.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can compute and interpret the discriminant.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can determine the number of real solutions from the discriminant.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can choose the most efficient method (factoring, completing the square, formula).
Confident ☐ Mostly ☐ Need Review ☐

Unit 6 · Week 23

Applications of Quadratics

Apply quadratic functions to real life — find maximum profit, optimal area, peak heights, and break-even points.

LESSON OVERVIEW

Welcome to Week 23. This week we apply quadratic functions to real-world situations.

Quadratic functions model a remarkable range of phenomena: projectile motion, profit maximization, area problems, and break-even points.

The vertex of a parabola has special meaning in applications. If the parabola opens downward (a is negative), the vertex is the maximum. If it opens upward (a is positive), the vertex is the minimum.

For projectile motion, the height equation often has the form h equals negative x squared plus some linear term. The vertex tells us when and how high the object reaches its peak.

For example, a ball follows h equals negative x squared plus 6 x. The axis is x equals 3. The max height is h of 3 equals 9.

For profit optimization, a business profit equation might be P equals negative x squared plus 10 x plus 24. The vertex gives the maximum profit.

For area problems, if a rectangle has a fixed perimeter of 20, with width x and length 10 minus x, the area is x times (10 minus x), or negative x squared plus 10 x. The max area occurs at x equals 5.

The x-intercepts of an application function have meaning too: they show when something equals zero — when a ball hits the ground, when profit breaks even, or when a quantity is depleted.

Always interpret your answers in context. A negative time or negative quantity may be mathematically correct but physically meaningless — reject those solutions.

KEY CONCEPTS & DID YOU KNOW

★ KEY FORMULA

Vertex is maximum if $a < 0$, minimum if $a > 0$

💡 **DID YOU KNOW?**

When the Wright brothers designed their famous aircraft, they used quadratic equations to model the lift and drag forces on their wings. Modern aerospace engineers still rely on quadratic models today.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1	**Projectile: $h = -x^2 + 6x$. Max height?**
Step 1	$a = -1$, $b = 6$
Step 2	Axis: $x = -6/(2\cdot -1) = 3$
Step 3	$h(3) = -9 + 18 = 9$
ANSWER	**Max height 9 at x = 3**

EXAMPLE 2	**Profit: $P = -x^2 + 10x + 24$. Max profit?**
Step 1	Axis: $x = -10/(-2) = 5$
Step 2	$P(5) = -25 + 50 + 24 = 49$
ANSWER	**Max profit 49 at x = 5**

EXAMPLE 3	**Rectangle, perimeter 20. Max area?**
Step 1	Dimensions: x and $10 - x$
Step 2	Area: $A = x(10 - x) = -x^2 + 10x$
Step 3	Axis: $x = 5$
Step 4	$A(5) = 25$
ANSWER	**Max area 25 at x = 5**

EXAMPLE 4	**$y = -2x^2 + 8x + 3$. Vertex?**

Step 1	$a=-2$, $b=8$
Step 2	$x = -8/(2\cdot -2) = 2$
Step 3	$y = -8 + 16 + 3 = 11$
ANSWER	**Max value 11 at $x = 2$**

EXAMPLE 5 Break-even: $P = x^2 - 5x - 14$. Find positive x.

Step 1	Factor: $(x - 7)(x + 2) = 0$
Step 2	$x = 7$ or $x = -2$
Step 3	Only positive: $x = 7$
ANSWER	**Break-even at $x = 7$**

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 A ball: $h = -x^2 + 4x$. Find maximum height.

2 Profit function: $P = -x^2 + 6x + 10$. Find maximum profit.

3 A rectangle has area $A = -x^2 + 12x$. Find maximum area.

4 Solve break-even: $x^2 - 9x + 14 = 0$.

5 Interpret vertex of $y = -3x^2 + 12x - 5$.

6

A rectangle has perimeter 20. Find dimensions for maximum area.

7

A ball: $h = -x^2 + 10x$. When does it hit the ground?

8

A ball: $h = -x^2 + 10x$. Find maximum height.

9 Profit $P = -2x^2 + 12x - 10$. Find maximum.

10 A rectangle has perimeter 16. Maximum area?

11 $h = -x^2 + 8x + 3$. Find when $h = 0$ (lands).

12 Profit: $P = x^2 - 5x - 14$. Find break-even points.

13 A javelin: $h = -x^2 + 12x$. Find max height and time at max.

14 Find break-even: $x^2 - 11x + 28 = 0$.

15

Maximum value of $y = -2x^2 + 8x + 3$?

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. For $y = -x^2 + 4x$, max value is at x =
- **A.** 1
- **B.** 2
- **C.** 4
- **D.** 0

2. A ball follows $h = -x^2 + 10x$. Maximum height equals:
- **A.** 10
- **B.** 25
- **C.** 100
- **D.** 20

3. If $a < 0$, the vertex is a:
- **A.** Minimum
- **B.** Maximum
- **C.** x-intercept
- **D.** y-intercept

4. A rectangle has perimeter 16. Max area:
- **A.** 16
- **B.** 64
- **C.** 16^2
- **D.** 64^2

5. For profit $P = -2x^2 + 12x - 10$, max profit at x =
- **A.** 12
- **B.** 3
- **C.** −3
- **D.** 6

6. Break-even points are found by setting:
- **A.** Profit = max
- **B.** Profit = 0
- **C.** x = 0
- **D.** Profit = constant

7. Quadratics best model situations involving:
- **A.** Constant growth
- **B.** Constant rate
- **C.** Curves and peaks

- **D.** Straight lines only

8. In application problems, a negative time solution should be:
- **A.** Accepted
- **B.** Rejected as unrealistic
- **C.** Squared
- **D.** Used anyway

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can interpret the vertex as a maximum or minimum in real-world problems.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can solve projectile motion problems with quadratic functions.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can find maximum area or maximum profit using quadratics.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can find break-even points by setting profit to zero.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can reject physically unrealistic solutions.
Confident ☐ Mostly ☐ Need Review ☐

Unit 6 Answer Key

Compare your answers below. If you missed a question, review the corresponding lesson section.

Week 19: Graphing Quadratics

Q#	Answer	Explanation
1	**B**	$x = -b/(2a) = -(-6)/2(1) = 3$.
2	**B**	Positive a → opens upward (smile shape).
3	**B**	$a = -3 < 0$ → opens downward.
4	**A**	Vertex form: $y = a(x-h)^2+k$ has vertex $(h, k) = (4, 2)$.
5	**C**	Set $x = 0$: $y = c = -7$.
6	**B**	$x^2 = 25$, so $x = \pm5$.
7	**B**	Larger $\|a\|$ means narrower parabola.
8	**B**	The axis of symmetry always passes through the vertex.

Week 20: Solving Quadratics by Factoring

Q#	Answer	Explanation
1	**B**	$(x+2)(x+5) = 0$, so $x = -2$ or -5.
2	**B**	Difference of squares: $(x-3)(x+3) = 0$, $x = \pm3$.
3	**B**	If a product equals zero, at least one factor must be zero.
4	**A**	$(2x+1)(x+2)=0$, so $x = -1/2$ or -2.
5	**B**	$3x(x+4) = 0$, so $x = 0$ or -4.
6	**A**	$(x-2)(x-4) = 0$, so $x = 2$ or 4.
7	**B**	Move all terms to one side so the equation equals 0.
8	**C**	$-x^2 + 6x = 0 \rightarrow x(-x+6)=0 \rightarrow x = 0$ (start) or $x = 6$ (lands).

Week 21: Completing the Square

Q#	Answer	Explanation
1	C	Half of 8 is 4; $4^2 = 16$.
2	B	Half of −10 is −5; $(-5)^2 = 25$.
3	B	Vertex form: $y = a(x - h)^2 + k$ with vertex (h, k).
4	A	Vertex form: (h, k) = (3, 5).
5	C	To keep the equation balanced, add to both sides.
6	B	$9 = 3^2$, middle term 6 = 2(3), so $(x + 3)^2$.
7	A	Add and subtract 4: $y = (x^2 + 4x + 4) - 4 = (x+2)^2 - 4$.
8	B	h = −2 (from x − h = x + 2), k = −4.

Week 22: The Quadratic Formula

Q#	Answer	Explanation
1	B	$x = (-b \pm \sqrt{(b^2 - 4ac)})/(2a)$.
2	B	The discriminant is $b^2 - 4ac$.
3	C	Positive discriminant → two distinct real solutions.
4	B	Zero discriminant → one (repeated) real solution.
5	A	Match to $ax^2 + bx + c$: a=3, b=−6, c=2.
6	A	Disc: 49−40 = 9. x = (−7 ± 3)/2 = −2 or −5.
7	C	Disc: 4 − 40 = −36 < 0 → no real solutions.
8	C	It works for ANY quadratic equation in standard form.

Week 23: Applications of Quadratics

Q#	Answer	Explanation
1	B	Axis: x = −4/(−2) = 2.
2	B	Axis x = 5, h(5) = −25 + 50 = 25.
3	B	a < 0 → opens downward → vertex is at top → maximum.

Q#	Answer	Explanation
4	**A**	Width x, length 8−x. A = x(8−x), max at x=4, A=16.
5	**B**	x = −12/(−4) = 3.
6	**B**	Break-even means profit = 0.
7	**C**	Quadratics handle curves: projectiles, profits, optimization.
8	**B**	Time cannot be negative — reject such answers in context.

UNIT 7

Exponential & Logarithmic Functions

WEEKS COVERED

Unit 7 · Week 24

Exponential Functions

Meet the explosive growth and rapid decay of exponential functions — modelling everything from compound interest to the spread of news through a village.

LESSON OVERVIEW

Welcome to Unit 7. This week we begin our study of exponential functions — perhaps the most powerful tool in modern algebra.

An exponential function has the form y equals a times b to the x. The base b is fixed, while the exponent x is the variable.

When the base b is greater than 1, we have exponential growth — the function increases faster and faster.

When b is between 0 and 1, we have exponential decay — the function decreases rapidly, then more slowly.

A common form for real-world growth is y equals a times the quantity 1 plus r, raised to x. Here a is the starting value and r is the growth rate.

For example, a city of 1000 people growing at 5 percent per year follows y equals 1000 times 1.05 to the x.

For decay, the form is y equals a times the quantity 1 minus r, raised to x. A 50 milligram drug that decreases 20 percent per hour follows y equals 50 times 0.8 to the x.

Exponential graphs all share a key feature: a horizontal asymptote at y equals zero. The graph approaches but never touches the x-axis.

Exponential change is fundamentally different from linear. A linear function adds the same amount each step. An exponential function multiplies by the same factor each step.

By the end of today, you will identify, evaluate, and graph exponential functions and distinguish growth from decay.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

Growth: $y = a(1 + r)^x$ · **Decay:** $y = a(1 - r)^x$

💡 **DID YOU KNOW?**

Compound interest — the engine of exponential financial growth — was famously called by Albert Einstein "the eighth wonder of the world." Whether he actually said it is disputed, but the math is real.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1 **Evaluate $y = 2(3)^x$ at $x = 2$**	
Step 1	$y = 2 \times 3^2$
Step 2	$= 2 \times 9$
Step 3	$= 18$
ANSWER	**$y = 18$**

EXAMPLE 2 **Growth or decay? $y = 5(0.7)^x$**	
Step 1	Identify base: 0.7
Step 2	$0 < 0.7 < 1 \rightarrow$ decay
ANSWER	**Exponential decay**

EXAMPLE 3 **Graph $y = 2^x$ — fill table**	
Step 1	$x=-2$: $y=1/4$
Step 2	$x=-1$: $y=1/2$
Step 3	$x=0$: $y=1$
Step 4	$x=1$: $y=2$
Step 5	$x=2$: $y=4$
ANSWER	**Curve through (0,1), passing through (1,2)**

EXAMPLE 4 **City of 1000, grows 5%/year. After 3 years?**	
Step 1	Model: $y = 1000(1.05)^x$
Step 2	$x = 3$
Step 3	$y = 1000 \times 1.157625$
Step 4	≈ 1158
ANSWER	**≈ 1158 people**

EXAMPLE 5 **Drug starts at 50 mg, decays 20%/hour. After 2 hours?**	
Step 1	Model: $y = 50(0.8)^x$
Step 2	$x = 2$
Step 3	$y = 50 \times 0.64$
Step 4	$= 32$
ANSWER	**32 mg**

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1

Evaluate: $y = 3(2)^x$ at $x = 3$

2

Evaluate: $y = 4(2)^x$ at $x = 4$

3

Identify growth or decay: $y = 10(0.6)^x$

4

Identify growth or decay: $y = 100(1.05)^x$

5

Graph $y = 4^x$ using table values for $x = -2, -1, 0, 1, 2$.

6

Write a model: 2000 grows at 3% per year

7

A substance decays 15% per hour from 80 units. Find value after 2 hours.

8

A car loses 12% value per year, starting at $15,000. Write a model.

9 Evaluate: $y = 5(2)^x$ at $x = 0$

10 Identify growth or decay: $y = 50(0.8)^x$

11 A population of 500 grows at 8% annually. Write a model.

12 What is the horizontal asymptote of $y = 7(0.5)^x$?

13 Evaluate: $y = 2(3)^x$ at $x = 4$

14 A bacteria population triples every hour, starting at 50. Write a model for time x.

15

For $y = 1000(1.04)^x$, what does 1.04 represent?

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. For $y = 3(4)^x$, find y when x = 2

- **A.** 12
- **B.** 24
- **C.** 48
- **D.** 36

2. $y = 100(2)^x$ at x = 0 equals:

- **A.** 0
- **B.** 1
- **C.** 100
- **D.** 200

3. $y = 5(0.9)^x$ represents:

- **A.** Growth
- **B.** Decay
- **C.** Linear
- **D.** Constant

4. $y = 100(1.08)^x$ represents:

- **A.** Growth
- **B.** Decay
- **C.** Linear
- **D.** Constant

5. Horizontal asymptote of $y = 2^x$:

- **A.** y = 2
- **B.** y = 0
- **C.** y = 1
- **D.** x = 0

6. Population starts at 500, grows 10%/yr. Model:

- **A.** y = 500 + 10x
- **B.** $y = 500(1.10)^x$
- **C.** $y = 500(0.10)^x$
- **D.** y = 500 × 10x

7. A car loses 15% value per year. Model with initial $20000:

- **A.** $y = 20000(0.85)^x$
- **B.** $y = 20000(1.15)^x$
- **C.** y = 20000 − 15x

- **D.** $y = 20000(15)^x$

8. Compared to linear $y = 100 + 50x$, the function $y = 100(1.5)^x$:

- **A.** Grows slower
- **B.** Grows faster after some time
- **C.** Grows the same
- **D.** Decays

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can identify exponential functions in the form $y = ab^x$.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can distinguish between exponential growth and decay from the base.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can evaluate exponential functions.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can write an exponential model from a real-world situation.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can describe the horizontal asymptote of an exponential graph.
Confident ☐ Mostly ☐ Need Review ☐

Unit 7 · Week 25

Applications of Exponential Models

Apply exponential models to compound interest, population, radioactive decay, and the doubling of bacteria — each governed by the same elegant formula.

LESSON OVERVIEW

Welcome to Week 25. Today we apply exponential models to real-world situations.

The compound interest formula is among the most useful. A equals P times the quantity 1 plus r, raised to t. Here P is the principal, r is the annual interest rate, and t is time in years.

For example, 1000 birr invested at 5 percent for 3 years: A equals 1000 times 1.05 cubed, which is approximately 1157 birr 63 cents.

Population growth follows a similar formula. A city of 50,000 growing at 2 percent per year has population A equals 50000 times 1.02 to the t.

For radioactive decay or substances that diminish over time, we use the form A equals starting amount times (1 minus rate) to the time.

For doubling problems, like a bacteria population that doubles every 3 hours starting at 100: A equals 100 times 2 to the t over 3 power. After 6 hours, t over 3 equals 2, so A equals 100 times 4, equals 400 bacteria.

For continuous growth — used in advanced finance and biology — we use the formula A equals P times e to the rt, where e is Euler's number, approximately 2.718.

For example, 500 birr growing at 6 percent continuously for 4 years: A equals 500 times e to the 0.24, approximately 635 birr 41 cents.

Choose the appropriate formula based on whether growth is discrete (yearly compounding) or continuous (compounded every instant).

By the end of today, you will model and solve real-world exponential problems involving interest, population, and decay.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

$A = P(1 + r)^t$ · Continuous: $A = Pe^{rt}$ · Doubling: $A = a(2)^{(t/d)}$

💡 **DID YOU KNOW?**

The number e (approximately 2.71828) appears throughout nature — in the growth of bacteria, the cooling of coffee, and the decay of radioactive atoms. It was first studied by Swiss mathematician Jacob Bernoulli in 1683.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1 1000 birr at 5% for 3 years

Step 1	$A = P(1+r)^t$
Step 2	$A = 1000(1.05)^3$
Step 3	$A = 1000 \times 1.157625$
Step 4	$A \approx 1157.63$

ANSWER ≈ 1157.63 birr

EXAMPLE 2 City: 50000 people, 2%/yr growth, 5 years

Step 1	$A = 50000(1.02)^5$
Step 2	$A \approx 50000 \times 1.1041$
Step 3	$A \approx 55{,}204$

ANSWER ≈ 55,204 people

EXAMPLE 3 Substance: 200 g, decays 10%/hr, 4 hours

Step 1	$A = 200(0.9)^4$
Step 2	$A \approx 200 \times 0.6561$
Step 3	$A \approx 131.22$ g

ANSWER ≈ 131.22 g

EXAMPLE 4 **Bacteria: doubles every 3 hrs, start 100, after 6 hrs**

Step 1	$A = 100(2)^{t/3}$
Step 2	$t = 6$, so $t/3 = 2$
Step 3	$A = 100 \times 4$

ANSWER **400 bacteria**

EXAMPLE 5 **Continuous: 500 birr at 6%, 4 years**

Step 1	$A = Pe^{rt}$
Step 2	$A = 500e^{(0.06 \times 4)}$
Step 3	$A = 500e^{0.24}$
Step 4	$A \approx 635.41$

ANSWER **≈ 635.41 birr**

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1

$1500 invested at 4% for 5 years (compound interest)

2

$1000 invested at 5% for 3 years

3

$2500 invested at 6% for 4 years

4

Population of 20,000 grows at 3% for 6 years

5

Population of 50,000 grows at 2% for 5 years

6

300 grams decays at 8% per hour for 3 hours

7

200 grams decays at 10% per hour for 4 hours

8

Bacteria doubles every 2 hours starting at 50. After 6 hours?

9 Bacteria doubles every 3 hours starting at 100. After 9 hours?

10 $800 grows continuously at 5% for 3 years.

11 $500 grows continuously at 6% for 4 years.

12 A substance halves every 5 hours. Starting at 1000 g, find amount after 10 hrs.

13 Drug dose of 60 mg decays 25% per hour. After 3 hours?

14 A city of 80,000 declines 2% per year. After 10 years?

15

How long for $1000 to double at 7% compound annual rate? (Estimate)

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. Compound interest formula:
- **A.** $A = P + rt$
- **B.** $A = P(1+r)^t$
- **C.** $A = Pr/t$
- **D.** $A = P^t$

2. Continuous growth formula:
- **A.** A = Pe^rt
- **B.** $A = P(1+r)^t$
- **C.** $A = Prt$
- **D.** A = Pe^t

3. $2000 at 4% for 5 years (compound, annual):
- **A.** $2400
- **B.** $2160.32
- **C.** $2433.31
- **D.** $2500

4. A population doubles every 5 years. After 10 years, factor:
- **A.** 2
- **B.** 4
- **C.** 10
- **D.** 5

5. Substance halves every 4 hours. After 8 hours, fraction left:
- **A.** 1/2
- **B.** 1/4
- **C.** 1/8
- **D.** 1/16

6. In $y = a(1 - r)^t$, when $r = 0.20$:
- **A.** Growth 20%
- **B.** Decay 20%
- **C.** Growth 80%
- **D.** Decay 80%

7. Choose appropriate model for continuous interest:
- **A.** Linear
- **B.** $y = a + bt$
- **C.** $A = P(1+r)^t$

D. A = Pe^(rt)

8. In the formula A = 100(2)^(t/3), the doubling time is:

A. 2 hours
B. 3 hours
C. 100 hours
D. 6 hours

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can apply the compound interest formula $A = P(1+r)^t$.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can model population growth and decay with exponential functions.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can solve doubling and half-life problems.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can use the continuous growth formula $A = Pe^{rt}$.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can choose the right exponential model for a given context.
Confident ☐ Mostly ☐ Need Review ☐

Unit 7 · Week 26

Introduction to Logarithms

Logarithms — the inverse of exponentials — answer the question: "What exponent do I need?"

LESSON OVERVIEW

Welcome to Week 26. This week we meet logarithms — one of the most powerful tools in mathematics.

A logarithm answers a single question: To what exponent must a base be raised to produce a given number?

In symbols: log base b of y equals x is the same as saying b to the x equals y. They are two ways of writing the same relationship.

For example, log base 2 of 8 equals 3, because 2 cubed equals 8.

Logarithms come in two especially common forms. The common logarithm uses base 10 — written as just "log." The natural logarithm uses base e, where e is Euler's number — written as "ln."

To evaluate log base 3 of 81, ask yourself: what power of 3 gives 81? Since 3 to the 4 is 81, the answer is 4.

Logarithms are only defined for positive numbers. You cannot take the log of zero or a negative number.

Logarithms appear everywhere in science. The Richter scale for earthquakes is logarithmic. The decibel scale for sound is logarithmic. The pH scale in chemistry is logarithmic.

These scales compress huge numbers into manageable ones. A magnitude 6 earthquake is 10 times stronger than a magnitude 5, but the logarithm scale shows just a 1-unit difference.

By the end of today, you will convert between exponential and logarithmic forms, and evaluate common and natural logarithms.

KEY CONCEPTS & DID YOU KNOW

★ KEY FORMULA

$\log_b(y) = x \Leftrightarrow b^x = y$ · log (base 10) · ln (base e)

💡 **DID YOU KNOW?**

Logarithms were invented in 1614 by Scottish mathematician John Napier to simplify multiplication. Before electronic calculators, scientists and engineers used log tables and slide rules to perform complex calculations.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1 Convert $2^3 = 8$ to log form

Step 1	Base: 2, exponent: 3, result: 8
Step 2	log_b(result) = exponent
Step 3	log_2(8) = 3

ANSWER $\log_2(8) = 3$

EXAMPLE 2 Convert $\log_5(125) = 3$ to exponential form

Step 1	log_b(y) = x → b^x = y
Step 2	$5^3 = 125$

ANSWER $5^3 = 125$

EXAMPLE 3 Evaluate $\log_3(81)$

Step 1	Ask: 3 to what power gives 81?
Step 2	$3^4 = 81$

ANSWER 4

EXAMPLE 4 Evaluate log(1000)

Step 1	Common log = base 10
Step 2	$10^3 = 1000$

ANSWER **3**

EXAMPLE 5 **Solve $\log_2(x) = 5$**

Step 1	Convert: $x = 2^5$
Step 2	$x = 32$

ANSWER **$x = 32$**

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 Convert to log form: $3^4 = 81$

2 Convert to log form: $2^5 = 32$

3 Convert to log form: $5^3 = 125$

4 Convert to exponential form: $\log_2(16) = 4$

5 Convert to exponential form: $\log_3(27) = 3$

6 Convert to exponential form: $\log(1000) = 3$

7 Evaluate: $\log_5(25)$

8 Evaluate: $\log(100)$

9 Evaluate: $\log_3(81)$

10 Evaluate: $\log(10{,}000)$

11 Evaluate: $\log_2(64)$

12 Evaluate: $\ln(e^5)$

13 Solve: $\log_3(x) = 2$

14 Solve: $\log_{10}(x) = 4$

15

Solve: $\log_2(x) = 5$

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. log_b(y) = x means:

- **A.** $x^y = b$
- **B.** $b^x = y$
- **C.** b + x = y
- **D.** y/x = b

2. Evaluate $\log_2(16)$

- **A.** 2
- **B.** 4
- **C.** 8
- **D.** 16

3. Evaluate log(100)

- **A.** 1
- **B.** 2
- **C.** 10
- **D.** 100

4. Evaluate $\log_5(25)$

- **A.** 1
- **B.** 2
- **C.** 5
- **D.** 25

5. Evaluate $\ln(e^3)$

- **A.** 1
- **B.** 3
- **C.** e
- **D.** e^3

6. Convert $3^2 = 9$ to log form:

- **A.** $\log_3(9) = 2$
- **B.** $\log_2(9) = 3$
- **C.** $\log_9(3) = 2$
- **D.** $\log_3(2) = 9$

7. Solve $\log_3(x) = 4$

- **A.** x = 7
- **B.** x = 12
- **C.** x = 64

- **D.** x = 81

8. Logarithms are defined for:

- **A.** All numbers
- **B.** Positive numbers only
- **C.** Negative numbers only
- **D.** Whole numbers only

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can define a logarithm and explain its relationship to exponents.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can convert between exponential and logarithmic forms.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can evaluate common logarithms (base 10).
Confident ☐ Mostly ☐ Need Review ☐

✓ I can evaluate natural logarithms (base e).
Confident ☐ Mostly ☐ Need Review ☐

✓ I can solve simple logarithmic equations using the definition.
Confident ☐ Mostly ☐ Need Review ☐

Unit 7 · Week 27

Solving Logarithmic Equations

Use the properties of logarithms — product, quotient, and power rules — to solve equations and unravel exponential mysteries.

LESSON OVERVIEW

Welcome to Week 27. This week we solve logarithmic equations using key properties.

There are three essential properties of logarithms. The product rule: log of x times y equals log x plus log y.

The quotient rule: log of x over y equals log x minus log y.

The power rule: log of x to the n equals n times log x. This rule is especially powerful — it lets us pull exponents out front.

To solve a logarithmic equation, we first combine all logs into a single log using these properties, then convert to exponential form to solve.

For example, solve log of x plus log of 3 equals 2. Apply the product rule: log of 3 x equals 2. Convert to exponential: 3 x equals 10 squared, so 3 x equals 100, and x equals 100 over 3.

When you have logs of the same base on both sides, you can simply set the arguments equal. For example, log base 3 of x equals log base 3 of 9 implies x equals 9.

Always check solutions in the original equation. Some answers may make a log undefined (taking the log of a negative number or zero) — these are extraneous and must be rejected.

For example, in log of x minus 1 equals log of 6, solving gives x equals 7. Check: x minus 1 equals 6, which is positive — valid.

By the end of today, you will solve logarithmic equations using all three log properties.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

Product: $\log(xy) = \log(x)+\log(y)$ · Power: $\log(x^n) = n\cdot\log(x)$

💡 **DID YOU KNOW?**

The power rule of logarithms — $\log(x^n) = n \cdot \log(x)$ — transformed astronomical calculations in the 1600s. Astronomers could replace tedious multiplications with simple additions, saving years of computational labor.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1 Solve $\log_2(x) = 4$

Step 1	Convert to exponential: $x = 2^4$
Step 2	$x = 16$

ANSWER $x = 16$

EXAMPLE 2 Solve $\log(x) + \log(3) = 2$

Step 1	Product rule: $\log(3x) = 2$
Step 2	Convert: $3x = 10^2 = 100$
Step 3	$x = 100/3$

ANSWER $x = 100/3$

EXAMPLE 3 Solve $\log(x) - \log(2) = 1$

Step 1	Quotient rule: $\log(x/2) = 1$
Step 2	Convert: $x/2 = 10$
Step 3	$x = 20$

ANSWER $x = 20$

EXAMPLE 4 Solve $\log_3(x) = \log_3(9)$

Step 1	Same base — set arguments equal

Step 2	$x = 9$
ANSWER	**$x = 9$**

EXAMPLE 5	**Solve $2 \log(x) = 4$**
Step 1	Power rule: $\log(x^2) = 4$
Step 2	Convert: $x^2 = 10^4 = 10000$
Step 3	$x = 100$ (reject negative)
ANSWER	**$x = 100$**

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1

Solve: $\log_2(x) = 3$

2

Solve: $\log_5(x) = 2$

3

Solve: $\log(x) + \log(4) = 2$

4

Solve: $\log(x) + \log(3) = 2$

5

Solve: $\log(x) - \log(5) = 1$

6

Solve: $\log(x) - \log(2) = 1$

7

Solve: $\log_3(x) = \log_3(27)$

8

Solve: $\log_4(x) = \log_4(16)$

9

Solve: $2 \log(x) = 6$

10

Solve: $2 \log(x) = 4$

11

Solve: $\log(x - 1) = \log(9)$

12

Solve: $\log_5(x) = 2$

13

Solve: $\log(x) + \log(2) = \log(10)$

14

Use product rule to expand: $\log(6x)$

15

Use quotient rule to expand: log(x/3)

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. Product rule: log(xy) =
- **A.** log(x) · log(y)
- **B.** log(x) + log(y)
- **C.** log(x) − log(y)
- **D.** log(x/y)

2. Power rule: $\log(x^n)$ =
- **A.** n + log(x)
- **B.** log(x)/n
- **C.** n · log(x)
- **D.** $\log(x)^n$

3. Quotient rule: log(x/y) =
- **A.** log(x) + log(y)
- **B.** log(x) − log(y)
- **C.** log(x) · log(y)
- **D.** log(xy)

4. Solve $\log_2(x) = 3$
- **A.** x = 2
- **B.** x = 6
- **C.** x = 8
- **D.** x = 32

5. Solve $\log_5(x) = 2$
- **A.** x = 5
- **B.** x = 10
- **C.** x = 25
- **D.** x = 50

6. Solve log(x) + log(4) = 2
- **A.** x = 2
- **B.** x = 25
- **C.** x = 50
- **D.** x = 100

7. $\log_3(x) = \log_3(27)$ implies:
- **A.** x = 3
- **B.** x = 9
- **C.** x = 27

- **D.** $x = 81$

8. After solving, must check that arguments are:

- **A.** Negative
- **B.** Zero
- **C.** Positive
- **D.** Whole numbers

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can apply the product, quotient, and power rules of logarithms.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can combine multiple logarithms into a single logarithm.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can convert a logarithmic equation to exponential form to solve.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can solve equations with logs on both sides.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can detect and reject extraneous logarithmic solutions.
Confident ☐ Mostly ☐ Need Review ☐

Unit 7 · Week 28

Applications of Logarithms

Logarithms in the wild — solve exponential equations, measure earthquake magnitudes, and find pH levels in chemistry.

LESSON OVERVIEW

Welcome to Week 28. This week we apply logarithms to real-world problems.

The most common use of logarithms is solving exponential equations where the unknown is in the exponent.

For example, solve 2 to the x equals 10. Take the log of both sides: log of 2 to the x equals log of 10. Using the power rule: x times log 2 equals 1. So x equals 1 over log 2, approximately 3.32.

For continuous growth equations like A equals P times e to the r t, we use natural logs to solve for time. If A equals 1000 times e to the 0.05 t equals 2000, divide by 1000, take ln of both sides: 0.05 t equals ln 2. So t equals ln 2 over 0.05, approximately 13.86.

The pH scale in chemistry uses logarithms. pH equals negative log of the hydrogen ion concentration. A concentration of 10 to the negative 3 gives pH equals 3 — acidic.

The decibel scale measures sound intensity. L equals 10 times log of intensity over reference intensity. A sound 1000 times the reference is 10 times log 1000, equals 30 decibels.

The Richter scale for earthquakes is also logarithmic. An earthquake 100 times stronger than another shows a magnitude difference of log 100, which is 2.

The change of base formula lets us evaluate any logarithm using common or natural logs: log base b of a equals log a over log b.

Whenever a problem asks for time or an unknown exponent in an exponential model, reach for logarithms.

By the end of today, you will solve exponential equations using logarithms and apply them to scientific scales.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

$a^x = b \Rightarrow x = \log_a(b)$ · Change of base: $\log_b(a) = \log(a)/\log(b)$

💡 **DID YOU KNOW?**

When 1980's Mount St. Helens erupted, it measured 5.1 on the Richter scale. The 1960 Valdivia earthquake measured 9.5 — but because the scale is logarithmic, the Valdivia quake released over 50,000 times more energy.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1	**Solve $2^x = 10$**
Step 1	Take log of both sides
Step 2	$x \log(2) = \log(10) = 1$
Step 3	$x = 1/\log(2) \approx 3.32$
ANSWER	**$x \approx 3.32$**

EXAMPLE 2	**A = 1000e^(0.05t) = 2000. Find t.**
Step 1	Divide by 1000: 2 = e^(0.05t)
Step 2	Take ln: $\ln 2 = 0.05t$
Step 3	$t = \ln 2 / 0.05$
Step 4	$t \approx 13.86$ years
ANSWER	**$t \approx 13.86$ years**

EXAMPLE 3	**Find pH when $[H^+] = 10^{-3}$**
Step 1	$pH = -\log[H^+]$
Step 2	$= -\log(10^{-3})$
Step 3	$= -(-3) = 3$
ANSWER	**pH = 3 (acidic)**

EXAMPLE 4 Sound: $I/I_0 = 1000$. Decibels?

Step 1	$L = 10 \log(I/I_0)$
Step 2	$L = 10 \log(1000)$
Step 3	$L = 10 \times 3$

ANSWER 30 dB

EXAMPLE 5 Solve $500(1.03)^t = 800$

Step 1	Divide: $(1.03)^t = 1.6$
Step 2	log both sides: $t \log(1.03) = \log(1.6)$
Step 3	$t = \log(1.6)/\log(1.03)$
Step 4	$t \approx 15.64$ years

ANSWER $t \approx 15.64$ years

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1

Solve: 3^x = 20

2

Solve: 2^x = 10

3

Solve: 5e^(0.04t) = 10

4

Find pH when $[H^+] = 10^{-5}$

5

Find pH when $[H^+] = 10^{-7}$

6

Find decibel level when intensity = 100 × reference

7

Find decibel level when intensity = 1000 × reference

8

Solve: 200(1.05)^t = 500

9

Solve: $500(1.03)^t = 800$

10

Solve: $10^x = 250$

11

Solve: $e^{(2x)} = 7$

12

An earthquake is 1000× stronger than another. Richter difference?

13

If pH = 3, what is $[H^+]$?

14

$A = 1000 e^{(0.05t)}$. When does A = 2000?

15

Solve: $4^x = 64$

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. To solve $3^x = 20$, you should:

- **A.** Take square root
- **B.** Take log of both sides
- **C.** Divide by x
- **D.** Multiply by 3

2. Change of base: log_b(a) =

- **A.** log(a) − log(b)
- **B.** log(b)/log(a)
- **C.** log(a)/log(b)
- **D.** log(a) · log(b)

3. pH formula:

- **A.** pH = log[H^+]
- **B.** pH = −log[H^+]
- **C.** pH = [H^+]
- **D.** pH = e^[H^+]

4. Decibel formula L =

- **A.** $\log(I/I_0)$
- **B.** $10 \log(I/I_0)$
- **C.** $I \times I_0$
- **D.** 10/I

5. An earthquake 1000× stronger has Richter difference:

- **A.** 1
- **B.** 2
- **C.** 3
- **D.** 1000

6. Solve $10^x = 250$ (approximately):

- **A.** x ≈ 2.40
- **B.** x ≈ 25
- **C.** x ≈ 250
- **D.** x ≈ 1

7. pH = 4 means [H^+] =

- **A.** 10^4
- **B.** 4
- **C.** 10^{-4}

- **D.** −4

8. Logarithmic scales compress:
- **A.** Small numbers
- **B.** Large numbers into manageable ranges
- **C.** Negative numbers
- **D.** Decimals

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can solve exponential equations by taking the logarithm of both sides.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can use the change of base formula.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can apply logarithms to pH, decibel, and Richter scale problems.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can solve continuous growth and decay equations for time.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can interpret logarithmic scales in real-world contexts.
Confident ☐ Mostly ☐ Need Review ☐

Unit 7 Answer Key

Compare your answers below. If you missed a question, review the corresponding lesson section.

Week 24: Exponential Functions

Q#	Answer	Explanation
1	**C**	$y = 3 \times 4^2 = 3 \times 16 = 48$.
2	**C**	Any number to the 0 power is 1, so $100 \times 1 = 100$.
3	**B**	Base $0.9 < 1 \rightarrow$ exponential decay.
4	**A**	Base $1.08 > 1 \rightarrow$ exponential growth.
5	**B**	All exponential functions $y = b^x$ have asymptote $y = 0$.
6	**B**	Growth: $a(1+r)^x = 500(1.10)^x$.
7	**A**	Decay: $a(1-r)^x = 20000(0.85)^x$.
8	**B**	Exponential growth eventually outpaces any linear growth.

Week 25: Applications of Exponential Models

Q#	Answer	Explanation
1	**B**	$A = P(1+r)^t$ for periodic compounding.
2	**A**	Continuous growth: A = Pe^(rt) where $e \approx 2.718$.
3	**C**	$A = 2000(1.04)^5 \approx 2000 \times 1.2167 \approx 2433.31$.
4	**B**	$t = 10$, $t/d = 2$, so $2^2 = 4\times$ original.
5	**B**	$8/4 = 2$ half-lives $\rightarrow (1/2)^2 = 1/4$.
6	**B**	$(1 - 0.20)^t = 0.8^t \rightarrow$ 20% decay per unit time.
7	**D**	Continuous: e^(rt) is the natural choice.
8	**B**	The denominator (3) is the doubling time.

Week 26: Introduction to Logarithms

Q#	Answer	Explanation
1	B	log_b(y) = x is the same as $b^x = y$.
2	B	$2^4 = 16$, so $\log_2(16) = 4$.
3	B	Common log: $10^2 = 100$.
4	B	$5^2 = 25$, so $\log_5(25) = 2$.
5	B	$\ln(e^3) = 3$ (natural log of e to a power gives the exponent).
6	A	Base 3, exponent 2, result 9 → $\log_3(9) = 2$.
7	D	$x = 3^4 = 81$.
8	B	log_b(y) requires y > 0.

Week 27: Solving Logarithmic Equations

Q#	Answer	Explanation
1	B	log of a product equals the sum of logs.
2	C	log of a power: the exponent comes out front.
3	B	log of a quotient equals the difference of logs.
4	C	$x = 2^3 = 8$.
5	C	$x = 5^2 = 25$.
6	B	log(4x) = 2 → 4x = 100 → x = 25.
7	C	Same base → arguments equal: x = 27.
8	C	Logarithms require positive arguments — reject if any log has a non-positive input.

Week 28: Applications of Logarithms

Q#	Answer	Explanation
1	B	Take log: x log(3) = log(20), then solve.
2	C	log_b(a) = log(a)/log(b).
3	B	$pH = -\log[H^+]$ (negative log of hydrogen concentration).

Q#	Answer	Explanation
4	**B**	$L = 10 \log(I/I_0)$.
5	**C**	log(1000) = 3 → 3 magnitudes higher.
6	**A**	$x = \log(250) \approx 2.40$.
7	**C**	$pH = -\log[H^+] = 4 \rightarrow [H^+] = 10^{-4}$.
8	**B**	Logarithms compress orders of magnitude into single-digit values.

UNIT 8

Systems of Equations & Inequalities

WEEKS COVERED

Week 29: Systems in Two Variables
Week 30: Systems in Three Variables
Week 31: Applications of Systems

Unit 8 · Week 29

Systems in Two Variables

Two equations, two unknowns — find the unique point where two relationships meet, whether by graphing, substitution, or elimination.

LESSON OVERVIEW

Welcome to Unit 8. This week we begin systems of equations — sets of equations that must all be true at the same time.

A system of equations involves two or more equations that share variables. The solution is the value or values that satisfy every equation in the system.

For two linear equations in x and y, the solution is the point (x, y) where both lines cross when graphed.

There are three ways to solve systems. The first is graphing. Plot both equations on the same axes and find their intersection point.

The second method is substitution. Solve one equation for one variable, then substitute that expression into the other equation.

The third method is elimination. Add or subtract the equations to eliminate one variable, then solve for the other.

Systems can have three different types of outcomes. A consistent system has one or more solutions. An inconsistent system has no solutions — this happens when the lines are parallel.

A dependent system has infinitely many solutions — this happens when the two equations represent the same line.

For example, y equals 3x plus 1 and y equals 3x minus 4 have the same slope but different intercepts. They are parallel, so no solution exists.

For example, 2x plus y equals 6 and 4x plus 2y equals 12 are the same line — infinitely many solutions.

By the end of today, you will solve systems using all three methods and identify which type of system you have.

KEY CONCEPTS & DID YOU KNOW

★ KEY FORMULA

Three methods: Graph · Substitute · Eliminate

💡 **DID YOU KNOW?**

The substitution and elimination methods were known to Chinese mathematicians by the year 200 BCE. Their text "Nine Chapters on the Mathematical Art" contains worked examples of systems of equations solved exactly as we do today.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1	**Graph: $y = 2x + 1$ and $y = -x + 4$**
Step 1	Plot first line through (0,1) with slope 2
Step 2	Plot second line through (0,4) with slope -1
Step 3	Find intersection visually
ANSWER	**Solution (1, 3)**

EXAMPLE 2	**Substitute: $x + y = 10$, $y = 2x$**
Step 1	Substitute $y = 2x$ into first: $x + 2x = 10$
Step 2	Combine: $3x = 10$
Step 3	Solve: $x = 10/3$
Step 4	Find y: $y = 20/3$
ANSWER	**(10/3, 20/3)**

EXAMPLE 3	**Eliminate: $2x + y = 7$, $2x - y = 3$**
Step 1	Add the two equations: $4x = 10$
Step 2	Solve: $x = 5/2$
Step 3	Substitute into first: $5 + y = 7$
Step 4	$y = 2$
ANSWER	**(5/2, 2)**

EXAMPLE 4 **No solution? $y = 3x + 1$, $y = 3x - 4$**

Step 1	Compare slopes: both 3
Step 2	Compare intercepts: 1 and −4 — different
Step 3	Lines are parallel

ANSWER **No solution (inconsistent)**

EXAMPLE 5 **Word problem: two numbers sum to 20, differ by 4**

Step 1	$x + y = 20$, $x - y = 4$
Step 2	Add: $2x = 24$, $x = 12$
Step 3	Substitute: $y = 8$

ANSWER **12 and 8**

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1

Solve: $y = x + 2$, $y = -x + 6$

2

Solve by substitution: $x + y = 12$, $y = 3x$

3

Solve by elimination: $3x + y = 11$, $3x - y = 5$

4

Identify type: $y = 4x + 1$, $y = 4x - 3$

5

Solve: $2x + 3y = 13$, $x - y = 1$

6

Word: two numbers sum to 30 and differ by 6. Find them.

7

Solve: $5x - 2y = 4$, $3x + 2y = 16$

8

Solve: $x + 2y = 10$, $3x - y = 9$

9

Identify type: $2x + y = 6$, $4x + 2y = 12$

10

Word: two numbers sum to 20 and differ by 4. Find them.

11

Solve graphically: $y = 2x + 1$ and $y = -x + 4$

12

Solve: $x - y = 2$, $2x + y = 7$

13

Solve: $4x + y = 13$, $2x + 3y = 9$

14

Word: Adult tickets cost \$8, child tickets \$5. 100 tickets sold for \$620. How many of each?

15

Solve: $6x + y = 11$, $x - y = 3$

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. In a system of two linear equations, "solution" means:
- **A.** Where lines meet
- **B.** y-intercept
- **C.** Slope
- **D.** Sum

2. Parallel lines indicate:
- **A.** One solution
- **B.** No solution
- **C.** Infinite solutions
- **D.** Two solutions

3. Identical lines indicate:
- **A.** No solution
- **B.** One solution
- **C.** Infinitely many solutions
- **D.** Two solutions

4. Solve $y = x + 2$, $y = -x + 6$
- **A.** (2, 4)
- **B.** (−2, 4)
- **C.** (4, 2)
- **D.** (0, 6)

5. Solve by elimination: $3x + y = 11$, $3x - y = 5$
- **A.** (8/3, 3)
- **B.** (8/3, −3)
- **C.** (3, 8/3)
- **D.** (2, 5)

6. Two numbers sum to 30 and differ by 6. They are:
- **A.** 10, 20
- **B.** 12, 18
- **C.** 15, 15
- **D.** 13, 17

7. For $y = 4x + 1$ and $y = 4x - 3$, the system is:
- **A.** Consistent
- **B.** Inconsistent
- **C.** Dependent

D. Unsolvable

8. Best method when one equation is already solved for y:

A. Graphing
B. Substitution
C. Elimination
D. None

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can solve systems of two linear equations by graphing.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can solve systems using the substitution method.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can solve systems using the elimination method.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can identify consistent, inconsistent, and dependent systems.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can translate word problems into systems of equations.
Confident ☐ Mostly ☐ Need Review ☐

Unit 8 · Week 30

Systems in Three Variables

Three equations, three unknowns — extend your toolkit to a three-dimensional puzzle, reducing dimension by dimension.

LESSON OVERVIEW

Welcome to Week 30. Today we extend systems to three variables — x, y, and z.

A system in three variables has three equations with three unknowns. The solution is an ordered triple (x, y, z) that satisfies all three equations.

Geometrically, each equation represents a plane in three-dimensional space. The solution is where all three planes intersect.

The general strategy is to reduce the three-variable system to a two-variable system by eliminating one variable.

Pick a variable to eliminate. Combine two equations to remove it. Then combine a different pair to remove it again. You now have two equations in two variables.

Solve that smaller system using methods you already know. Then substitute back to find the third variable.

For example, with x plus y plus z equals 6, two x plus y minus z equals 3, and x minus y plus 2 z equals 5: eliminate z, solve the resulting 2-variable system, then substitute.

When equations contradict each other — like x plus y plus z equals 5 AND x plus y plus z equals 7 — the system is inconsistent. No solution.

When one equation is just a scalar multiple of another, the system is dependent — infinitely many solutions.

Always verify your solution by substituting back into all three original equations.

By the end of today, you will solve three-variable systems using elimination and substitution.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

Eliminate one variable to get a 2-variable system, then solve

💡 **DID YOU KNOW?**

Three-variable systems are the foundation of modern computer graphics. When your phone renders a 3D image, it solves thousands of three-variable systems every second to determine where each pixel should be.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1 Solve 3-var: x+y+z=6, 2x+y−z=3, x−y+2z=5

Step 1	Add (1)+(2): 3x + 2y = 9
Step 2	Multiply (1) by 2, subtract (3): x + 3y = 7
Step 3	Solve 2-var system: 3x+2y=9, x+3y=7
Step 4	Solve: y=12/7, x=13/7
Step 5	Substitute: z = 17/7

ANSWER (13/7, 12/7, 17/7)

EXAMPLE 2 Substitution: x+y+z=9, y=x+1, z=2

Step 1	Plug into first: x + (x+1) + 2 = 9
Step 2	Simplify: 2x + 3 = 9
Step 3	Solve: x = 3
Step 4	Find y: 4

ANSWER (3, 4, 2)

EXAMPLE 3 No solution check: x+y+z=5, x+y+z=7

Step 1	Same left side, different right side
Step 2	Impossible contradiction

ANSWER No solution

EXAMPLE 4 Dependent check: $x+y+z=6$, $2x+2y+2z=12$

Step 1	Divide second by 2: $x+y+z=6$
Step 2	Same equation!
Step 3	Dependent

ANSWER Infinitely many solutions

EXAMPLE 5 Verify (2,1,3) for $x+y+z=6$, $2x+y-z=2$, $x-y+z=4$

Step 1	$2+1+3=6$ ✓
Step 2	$4+1-3=2$ ✓
Step 3	$2-1+3=4$ ✓

ANSWER Valid solution

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 Solve: $x + y + z = 6$, $2x + y - z = 4$, $x - y + 2z = 5$

2 Solve: $x + y + z = 10$, $y = 2x$, $z = 3$

3 Determine if solvable: $x + y + z = 3$, $x + y + z = 5$, $x - y + z = 1$

4 Solve: $2x + y + z = 8$, $x + 2y + z = 9$, $x + y + 2z = 10$

5 Word: three numbers sum to 18, second is twice first, third is first plus 3. Find them.

6

Solve: $x + 2y + z = 7$, $2x + y + z = 8$, $x + y + 2z = 9$

7

Verify (1,2,3) in: $x + y + z = 6$, $2x + y - z = 1$, $x - y + z = 2$

8

Solve: $x + y + z = 5$, $2x - y + z = 4$, $x + y - z = 1$

9

Solve: $x + 2y - z = 4$, $3x - y + z = 5$, $2x + y + z = 7$

10

Verify (0, 3, 1) in: $x + y + z = 4$, $2x + y - z = 2$

11

Solve: $x = 5$, $y + z = 4$, $y - z = 0$

12

Solve: $x + y = 7$, $y + z = 9$, $x + z = 8$

13

Determine type: $2x + 2y + 2z = 10$, $x + y + z = 5$, $x + y - z = 1$

14

Word: a store sells pens, notebooks, folders. Total 30 items, $2x + y + 3z = 70$, $x + 2y + z = 40$.

15 Solve: $x - y + 2z = 4$, $2x + y - z = 1$, $x + 2y + z = 7$

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. A solution to a 3-variable system is:

- **A.** A single number
- **B.** An ordered pair
- **C.** An ordered triple
- **D.** A line

2. Strategy: convert 3-variable system to:

- **A.** 1-variable
- **B.** 2-variable
- **C.** 4-variable
- **D.** Polynomial

3. x+y+z=5 and x+y+z=8 represent:

- **A.** One solution
- **B.** No solution
- **C.** Infinite solutions
- **D.** Two solutions

4. If second equation equals 2× the first, system is:

- **A.** Inconsistent
- **B.** Independent
- **C.** Dependent
- **D.** Solved

5. For (2, 1, 3) in equation $x + 2y - z = ?$, answer:

- **A.** 1
- **B.** 3
- **C.** 5
- **D.** 7

6. After finding x and y, you find z by:

- **A.** Guessing
- **B.** Substituting back into original equation
- **C.** Adding equations
- **D.** Squaring

7. Always verify a solution in:

- **A.** Just one equation
- **B.** Two equations
- **C.** All three equations

- **D.** Squared version

8. Three equations geometrically represent:
- **A.** Three points
- **B.** Three lines
- **C.** Three planes
- **D.** Three circles

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can solve systems of three linear equations by elimination.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can reduce a 3-variable system to a 2-variable system.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can identify consistent, inconsistent, and dependent 3-variable systems.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can verify a solution by substituting into all three equations.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can model real-world problems with three unknowns.
Confident ☐ Mostly ☐ Need Review ☐

Unit 8 · Week 31

Applications of Systems

Apply systems to mixture problems, rate problems, work problems, and business profit — wherever multiple unknowns interact.

LESSON OVERVIEW

Welcome to Week 31. This week we apply systems of equations to real-life problems.

Systems are powerful tools whenever a problem has multiple unknowns connected by multiple relationships.

The first major application is mixture problems. How many liters of one solution should be mixed with another to reach a target concentration?

Setup: x liters of solution A, y liters of solution B. Use two equations — one for total volume, one for total amount of solute. Then solve.

The second application is rate or distance problems. Two cars travel toward each other from different starts. When do they meet?

Use the formula distance equals rate times time. Set up an equation: the sum of both distances equals the total separation.

The third application is work problems. If one worker takes 6 hours and another takes 3, how long together? Use the combined work formula: 1 over T equals 1 over t one plus 1 over t two.

The fourth application is profit modeling. A company sells x units of one item and y units of another. Total units form one equation. Total profit forms the second.

The fifth application is supply and demand in economics. Equilibrium is where the supply curve and demand curve intersect — exactly a systems problem.

In every application: define your variables clearly, translate words into equations, solve systematically, and interpret your answer in context.

Reject answers that are physically unreasonable — negative time, negative quantities of items.

KEY CONCEPTS & DID YOU KNOW

★ KEY FORMULA

Define variables · Translate to equations · Solve ·

Verify in context

💡 **DID YOU KNOW?**

During the Apollo Moon missions, NASA used massive systems of equations to plot spacecraft trajectories — solving hundreds of equations simultaneously with the most advanced computers of the era.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1 **Mix 20% and 50% to get 10 L of 30%**	
Step 1	x = liters of 20%, y = liters of 50%
Step 2	x + y = 10
Step 3	0.2x + 0.5y = 3
Step 4	Solve: y = 10/3 ≈ 3.33
Step 5	x = 20/3 ≈ 6.67
ANSWER	**6.67 L of 20%, 3.33 L of 50%**

EXAMPLE 2 **Two cars 200 mi apart, 60 mph and 40 mph, time?**	
Step 1	Combined rate: 60 + 40 = 100 mph
Step 2	t = distance / rate
Step 3	t = 200/100 = 2 hours
ANSWER	**2 hours**

EXAMPLE 3 **Workers: 6 hours and 3 hours each. Together?**	
Step 1	1/T = 1/6 + 1/3
Step 2	= 1/6 + 2/6
Step 3	= 3/6 = 1/2
Step 4	T = 2 hours

ANSWER **2 hours**

EXAMPLE 4 **Profit: $x + y = 30$, $5x + 8y = 200$**

Step 1	From first: $x = 30 - y$
Step 2	Substitute: $5(30-y) + 8y = 200$
Step 3	$150 + 3y = 200$
Step 4	$y \approx 16.67$, $x \approx 13.33$

ANSWER **$x \approx 13.33$, $y \approx 16.67$ units**

EXAMPLE 5 **Supply $p = 2x + 10$, Demand $p = -x + 40$**

Step 1	Set equal: $2x + 10 = -x + 40$
Step 2	$3x = 30$, $x = 10$
Step 3	$p = 30$

ANSWER **Equilibrium (10, 30)**

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 Mix 10% and 40% solutions to get 20 liters of 25%. How much of each?

2 Mix 20% and 50% to get 10 liters of 30%. How much of each?

3 Two trains approach at 70 mph and 50 mph, 360 miles apart. Time to meet?

4 Two cars start 200 miles apart at 60 and 40 mph. Time to meet?

5 Workers: 4 and 12 hours. Time together?

6

Workers: 6 and 3 hours. Time together?

7

Profit: $x + y = 40$, $6x + 9y = 240$. Solve.

8

Profit: $x + y = 30$, $5x + 8y = 200$. Solve.

9

Supply: $p = 3x + 5$. Demand: $p = -2x + 50$. Equilibrium?

10

Supply: $p = 2x + 10$. Demand: $p = -x + 40$. Equilibrium?

11

Two numbers sum to 60, one is triple the other. Find them.

12

Two numbers sum to 50, one is 10 more than twice the other. Find them.

13

Combine 25% and 75% to get 24 L of 50%. Amounts?

14

Workers: 6 hours and 9 hours. Time together?

15 Two cars from same start, one at 50 mph and one at 70 mph (in same direction, 1 hr apart). Distance and time when second catches up?

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. In a mixture problem, two equations represent:

- **A.** Volume and concentration
- **B.** Volume only
- **C.** Concentration only
- **D.** Time and rate

2. Two cars approach each other at 70 mph and 50 mph, 360 mi apart. Time to meet:

- **A.** 3 hours
- **B.** 5 hours
- **C.** 7.2 hours
- **D.** 12 hours

3. Combined work formula 1/T =

- **A.** $t_1 + t_2$
- **B.** $1/t_1 + 1/t_2$
- **C.** $t_1 \times t_2$
- **D.** $t_1 - t_2$

4. Sum of two numbers is 60, one is triple the other. Smaller:

- **A.** 10
- **B.** 15
- **C.** 20
- **D.** 30

5. Workers: 4 and 12 hours each. Together?

- **A.** 16 hours
- **B.** 6 hours
- **C.** 3 hours
- **D.** 8 hours

6. Equilibrium occurs where:

- **A.** Supply > Demand
- **B.** Demand > Supply
- **C.** Supply = Demand
- **D.** Both = 0

7. A "negative number of items" solution should be:

- **A.** Used
- **B.** Rejected

- **C.** Squared
- **D.** Doubled

8. Mix 10% and 40% to get 20 L of 25%. How much 10%?

- **A.** 5 L
- **B.** 10 L
- **C.** 15 L
- **D.** 20 L

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can set up and solve mixture problems using systems.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can set up and solve distance-rate-time problems using systems.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can apply systems to combined work rate problems.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can solve profit, supply, and demand problems with systems.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can interpret solutions for real-world reasonableness.
Confident ☐ Mostly ☐ Need Review ☐

Unit 8 Answer Key

Compare your answers below. If you missed a question, review the corresponding lesson section.

Week 29: Systems in Two Variables

Q#	Answer	Explanation
1	**A**	The solution is the intersection point (x, y) satisfying both equations.
2	**B**	Same slope, different intercepts → no intersection.
3	**C**	Every point on the line satisfies both equations.
4	**A**	Set equal: $x + 2 = -x + 6$, so $2x = 4$, $x = 2$, $y = 4$.
5	**A**	Add: $6x = 16$, $x = 8/3$. Substitute: $y = 11 - 3(8/3) = 11 - 8 = 3$.
6	**B**	$x+y=30$, $x-y=6$. Add: $2x=36$, $x=18$, $y=12$.
7	**B**	Same slope, different intercepts → inconsistent (no solution).
8	**B**	Substitution is most efficient when y is already isolated.

Week 30: Systems in Three Variables

Q#	Answer	Explanation
1	**C**	Three variables → (x, y, z) ordered triple.
2	**B**	Eliminate one variable to reduce to a 2-variable system.
3	**B**	Contradictory equations → no solution.
4	**C**	Same line/plane → dependent, infinitely many solutions.
5	**A**	$2 + 2(1) - 3 = 2 + 2 - 3 = 1$.
6	**B**	Substitute known x and y into any original equation.
7	**C**	All three must be satisfied.
8	**C**	Each linear equation in x, y, z is a plane in 3D space.

Week 31: Applications of Systems

Q#	Answer	Explanation
1	**A**	One equation for total volume, one for total solute amount.
2	**A**	Combined rate 120 mph. Time = 360/120 = 3 hours.
3	**B**	$1/T = 1/t_1 + 1/t_2$.
4	**B**	$x + 3x = 60 \rightarrow x = 15$.
5	**C**	$1/T = 1/4 + 1/12 = 4/12 = 1/3$, so $T = 3$.
6	**C**	Equilibrium is where supply equals demand.
7	**B**	Cannot have negative count — reject as physically unreasonable.
8	**B**	$x + y = 20$, $0.1x + 0.4y = 5 \rightarrow x = 10$, $y = 10$.

UNIT 9

Sequences & Series

WEEKS COVERED

Week 32: Arithmetic Sequences

Week 33: Geometric Sequences

Week 34: Series and Applications

Unit 9 · Week 32

Arithmetic Sequences

Patterns that grow by a fixed amount each step — like saving the same number of birr in a clay pot every week, watching it stack up steadily.

LESSON OVERVIEW

Welcome to Unit 9. This week we begin our study of sequences — ordered lists of numbers that follow patterns.

An arithmetic sequence has a constant difference between consecutive terms. This fixed amount is called the common difference, denoted by d.

For example, 2, 5, 8, 11, 14. The common difference is 3 — we add 3 each time.

There are two ways to describe such a sequence. The explicit formula gives any term directly: a sub n equals a sub 1 plus n minus 1 times d, where a sub 1 is the first term.

For our example: a sub n equals 2 plus n minus 1 times 3. To find the 10th term: a sub 10 equals 2 plus 9 times 3, equals 29.

The recursive formula defines each term using the previous one: a sub n equals a sub n minus 1 plus d, with a starting value.

When we add up the terms of a sequence, we get a series. The sum of the first n terms of an arithmetic series is given by S sub n equals n over 2 times the sum of the first and last terms.

For example, sum of first 10 terms of 2, 5, 8, 11... The first term is 2, the 10th term is 29. So S equals 5 times 31, equals 155.

Arithmetic sequences model many real-life patterns: regular savings, weekly wages, linear depreciation.

By the end of today, you will identify arithmetic sequences, write their formulas, find any term, and compute sums.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

$a_n = a_1 + (n-1)d \cdot S_n = (n/2)(a_1 + a_n)$

💡 **DID YOU KNOW?**

The famous mathematician Carl Gauss, when he was just 7 years old, was given the problem of adding 1+2+3+...+100. He instantly realized the pairs (1+100), (2+99), (3+98)... each sum to 101. With 50 such pairs, the total is 5050.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1 Common difference of 2, 5, 8, 11

Step 1	$5 - 2 = 3$
Step 2	$8 - 5 = 3$
Step 3	$11 - 8 = 3$

ANSWER $d = 3$

EXAMPLE 2 Explicit formula for 4, 7, 10, 13

Step 1	$a_1 = 4, d = 3$
Step 2	$a_n = a_1 + (n-1)d$
Step 3	$a_n = 4 + (n-1)(3)$

ANSWER $a_n = 4 + 3(n - 1)$ or $3n + 1$

EXAMPLE 3 Find 10th term of 3, 6, 9, 12

Step 1	$a_1 = 3, d = 3$
Step 2	$a_{10} = 3 + (10-1)(3)$
Step 3	$a_{10} = 3 + 27$

ANSWER 30

EXAMPLE 4 Recursive formula for 5, 9, 13, 17

Step 1	$d = 4$
Step 2	$a_1 = 5$
Step 3	$a_n = a_{n-1} + 4$

ANSWER $a_1 = 5, a_n = a_{n-1} + 4$

EXAMPLE 5 Sum first 10 terms of 2, 5, 8, 11...

Step 1	$a_1 = 2, d = 3$
Step 2	$a_{10} = 2 + 9(3) = 29$
Step 3	$S_{10} = (10/2)(2 + 29)$
Step 4	$S_{10} = 5 \times 31$

ANSWER $S_{10} = 155$

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1

Find the common difference: 7, 10, 13, 16

2

Find the common difference: 2, 7, 12, 17

3

Write explicit formula: 6, 9, 12, 15

4

Write explicit formula: 4, 7, 10, 13

5

Find the 15th term: 2, 4, 6, 8

6

Find the 10th term: 3, 6, 9, 12

7

Find the 20th term: 5, 8, 11, 14

8

Write recursive formula: 1, 5, 9, 13

9

Write recursive formula: 8, 14, 20, 26

10

Find sum of first 12 terms: 3, 6, 9...

11

Find sum of first 10 terms: 2, 5, 8...

12

For $a_1 = 8$, $d = 2$, find a_{25}

13

For $a_1 = 5$, $d = 4$, find a_{15}

14

Real-life: weekly savings increases by $3 starting at $5. Find week 10 amount.

15

A student saves $20 in week 1 and $5 more each week. Total savings in 8 weeks?

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. Common difference of 7, 10, 13, 16:

- **A.** 2
- **B.** 3
- **C.** 4
- **D.** 7

2. Explicit formula for arithmetic sequence:

- **A.** $a_n = a_1 + n \cdot d$
- **B.** $a_n = a_1 + (n-1)d$
- **C.** $a_n = a_1 \cdot r^n$
- **D.** $a_n = a_1/n$

3. For $a_1 = 8$, $d = 2$, find a_5:

- **A.** 10
- **B.** 14
- **C.** 16
- **D.** 18

4. For 6, 9, 12, 15, the 8th term is:

- **A.** 24
- **B.** 27
- **C.** 30
- **D.** 33

5. Sum formula for arithmetic series:

- **A.** $S_n = n(a_1+a_n)$
- **B.** $S_n = (n/2)(a_1+a_n)$
- **C.** $S_n = a_1 + a_n$
- **D.** $S_n = n \cdot d$

6. Sum of first 5 terms of 1, 3, 5, 7, 9:

- **A.** 25
- **B.** 20
- **C.** 45
- **D.** 15

7. Recursive form requires:

- **A.** Only formula
- **B.** Starting value AND rule
- **C.** Just the rule

- **D.** Just a_1

8. Arithmetic sequences are best modeled by:
- **A.** Quadratics
- **B.** Linear functions
- **C.** Exponentials
- **D.** Logarithms

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can identify an arithmetic sequence and find its common difference.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can write the explicit formula for an arithmetic sequence.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can write the recursive formula for an arithmetic sequence.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can find any term in an arithmetic sequence.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can compute the sum of an arithmetic series.
Confident ☐ Mostly ☐ Need Review ☐

Unit 9 · Week 33

Geometric Sequences

Patterns that grow by a fixed factor — doubling, tripling, or halving — modelling everything from bacteria growth to compound interest.

LESSON OVERVIEW

Welcome to Week 33. Today we study geometric sequences — patterns where each term is multiplied by a fixed factor.

A geometric sequence has a constant ratio between consecutive terms. This ratio is called the common ratio, denoted by r.

For example, 3, 6, 12, 24, 48. Each term is 2 times the previous. So r equals 2.

The explicit formula is a sub n equals a sub 1 times r to the n minus 1.

For our example: a sub n equals 3 times 2 to the n minus 1. To find the 5th term: a sub 5 equals 3 times 2 to the 4, equals 48. ✓

The recursive formula is a sub n equals a sub n minus 1 times r, with a starting value.

When r is greater than 1, the sequence grows. When r is between 0 and 1, the sequence decays. When r is negative, signs alternate.

The sum of the first n terms of a geometric series uses a different formula: S sub n equals a sub 1 times the quantity 1 minus r to the n, all over 1 minus r.

For example, sum of first 5 terms of 2, 4, 8, 16, 32. Here $a_1 = 2$, r = 2, n = 5. S equals 2 times (1 − 32) over (1 − 2) equals 2 times 31, equals 62.

Geometric sequences appear in many real-life situations: population growth, compound interest, radioactive decay, bacteria multiplication.

By the end of today, you will identify geometric sequences, write formulas, and compute sums.

KEY CONCEPTS & DID YOU KNOW

★ KEY FORMULA

$$a_n = a_1 \cdot r^{n-1} \cdot S_n = a_1(1 - r^n)/(1 - r)$$

💡 DID YOU KNOW?

A classic geometric sequence story: a king offers a chess inventor any reward. The inventor asks for 1 grain on the first square, 2 on the second, 4 on the third — doubling each time. By the 64th square, the total exceeds 18 quintillion grains — more than all the wheat ever grown.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1 **Common ratio of 3, 6, 12, 24**

Step 1	$6 \div 3 = 2$
Step 2	$12 \div 6 = 2$
Step 3	$24 \div 12 = 2$

ANSWER $r = 2$

EXAMPLE 2 **Explicit formula for 5, 10, 20, 40**

Step 1	$a_1 = 5, r = 2$
Step 2	$a_n = a_1 \cdot r^{n-1}$
Step 3	$a_n = 5 \cdot 2^{n-1}$

ANSWER $a_n = 5 \cdot 2^{n-1}$

EXAMPLE 3 **Find 6th term of 2, 6, 18, 54**

Step 1	$a_1 = 2, r = 3$
Step 2	$a_6 = 2 \cdot 3^5$
Step 3	$a_6 = 2 \times 243$

ANSWER 486

EXAMPLE 4 **Recursive: 7, 21, 63, 189**

Step 1	$r = 3$
Step 2	$a_1 = 7$
Step 3	$a_n = a_{n-1} \cdot 3$

ANSWER $a_1 = 7, a_n = a_{n-1} \cdot 3$

EXAMPLE 5 **Sum first 5 of 2, 4, 8, 16, 32**

Step 1	$a_1 = 2, r = 2, n = 5$
Step 2	$S_5 = 2(1 - 32)/(1 - 2)$
Step 3	$= 2 \times (-31)/(-1)$
Step 4	$= 62$

ANSWER $S_5 = 62$

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 Find the common ratio: 4, 12, 36, 108

2 Find the common ratio: 5, 10, 20, 40

3 Find the common ratio: 81, 27, 9, 3

4 Write explicit formula: 3, 9, 27, 81

5 Write explicit formula: 5, 10, 20, 40

6

Find the 8th term: 2, 4, 8, 16

7

Find the 6th term: 2, 6, 18, 54

8

Write recursive formula: 6, 18, 54, 162

9

Write recursive formula: 100, 50, 25, 12.5

10

Find sum of first 6 terms: 1, 2, 4, 8...

11

Find sum of first 5 terms: 3, 6, 12, 24, 48

12

For $a_1 = 10$, $r = 3$, find a_5

13

For $a_1 = 2$, $r = 4$, find a_4

14

Real-life: a value triples each year starting at 50. Find year 5 value.

15

Bacteria starts at 100, doubles every hour. After 6 hours?

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. Common ratio of 4, 12, 36, 108:

- **A.** 2
- **B.** 3
- **C.** 4
- **D.** 12

2. Explicit formula for geometric sequence:

- **A.** $a_n = a_1 + (n-1)d$
- **B.** $a_n = a_1 \cdot r^{n-1}$
- **C.** $a_n = a_1 + r$
- **D.** $a_n = a_1 \times n$

3. For $a_1 = 10$, $r = 3$, $a_3 =$

- **A.** 30
- **B.** 60
- **C.** 90
- **D.** 100

4. For 3, 9, 27, 81, the 5th term is:

- **A.** 243
- **B.** 81
- **C.** 162
- **D.** 729

5. For 2, 4, 8, 16, the 8th term is:

- **A.** 64
- **B.** 128
- **C.** 256
- **D.** 512

6. When $0 < r < 1$, the sequence:

- **A.** Grows
- **B.** Decays
- **C.** Stays constant
- **D.** Alternates

7. Bacteria doubles each hour, starts at 100. After 5 hours:

- **A.** 500
- **B.** 1600
- **C.** 3200

- **D.** 100^2

8. A value triples each year starting at 50. Year 4:

- **A.** 150
- **B.** 450
- **C.** 1350
- **D.** 4050

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can identify a geometric sequence and find its common ratio.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can write the explicit formula for a geometric sequence.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can write the recursive formula for a geometric sequence.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can find any term in a geometric sequence.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can compute the sum of a finite geometric series.
Confident ☐ Mostly ☐ Need Review ☐

Unit 9 · Week 34

Series and Applications

Sums of sequences — finite and infinite, arithmetic and geometric — modelling savings totals, bouncing balls, and the elegant magic of convergence.

LESSON OVERVIEW

Welcome to Week 34. This week we explore series — the sums of sequences.

A sequence is a list of terms. A series is the sum of those terms.

We have two formulas for finite series. For arithmetic: S sub n equals n over 2 times the sum of the first and last terms.

For geometric: S sub n equals a sub 1 times the quantity 1 minus r to the n, all over 1 minus r.

There is also a special case — the infinite geometric series. When the absolute value of r is less than 1, the series converges to a finite sum.

The formula is S equals a sub 1 over the quantity 1 minus r.

For example, 5 plus 2.5 plus 1.25 plus dot dot dot. Here $a_1 = 5$ and $r = 0.5$. S equals 5 over (1 − 0.5), equals 5 over 0.5, equals 10.

Sigma notation is a compact way to write a sum. The Greek letter sigma, Σ, means "add up." We write the variable below (with start value), the end value above, and the term to the right.

For example, sigma from i equals 1 to 5 of 2 i means 2 plus 4 plus 6 plus 8 plus 10, which is 30.

Series appear in many real-life situations: total savings over time, the total distance a bouncing ball travels, financial annuities.

A bouncing ball dropped from 10 meters, bouncing to half its height each time, travels infinite bounces — but the total distance is finite: 30 meters.

By the end of today, you will compute arithmetic, geometric, and infinite geometric series, and use sigma notation.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

Arithmetic: $(n/2)(a_1+a_n)$ · Geometric: $a_1(1-r^n)/(1-r)$ · Infinite: $a_1/(1-r)$

💡 **DID YOU KNOW?**

The paradox of an infinite series with a finite sum confused even great thinkers like Zeno of Elea, who claimed that motion was impossible because crossing any distance required completing an infinite number of steps. Modern mathematics resolves this beautifully — convergence is real.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1	**Arithmetic series: sum first 10 of 3, 6, 9, 12...**
Step 1	$a_1 = 3, d = 3$
Step 2	$a_{10} = 3 + 9(3) = 30$
Step 3	$S_{10} = (10/2)(3 + 30)$
Step 4	$= 5 \times 33$
ANSWER	**$S_{10} = 165$**

EXAMPLE 2	**Geometric series: sum first 5 of 2, 4, 8, 16, 32**
Step 1	$a_1 = 2, r = 2, n = 5$
Step 2	$S_5 = 2(1 - 32)/(1 - 2)$
Step 3	$= 2 \times (-31)/(-1)$
ANSWER	**$S_5 = 62$**

EXAMPLE 3	**Infinite: 5 + 2.5 + 1.25 + ...**
Step 1	$a_1 = 5, r = 0.5$
Step 2	$\|r\| < 1$, converges
Step 3	$S = 5/(1 - 0.5)$
Step 4	$= 5/0.5$
ANSWER	**$S = 10$**

EXAMPLE 4 Sigma notation: Σ (i=1 to 5) 2i

Step 1	Expand: 2(1) + 2(2) + 2(3) + 2(4) + 2(5)
Step 2	= 2 + 4 + 6 + 8 + 10

ANSWER 30

EXAMPLE 5 Bouncing ball: 10 m, bounces to half each time. Total distance?

Step 1	Initial drop: 10 m
Step 2	Bounces (up + down): 2(5 + 2.5 + 1.25 + ...)
Step 3	Sum of infinite series: 5/(1 − 0.5) = 10
Step 4	Total: 10 + 2(10) = 30 m

ANSWER 30 meters

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 Find sum: 4 + 7 + 10 + ... (10 terms)

2 Find sum: 5 + 9 + 13 + ... (12 terms)

3 Find sum: 3 + 6 + 12 + 24 (6 terms, geometric)

4 Find sum: 2 + 6 + 18 + 54 + ... (5 terms)

5 Find infinite sum: 8 + 4 + 2 + 1 + ...

6

Find infinite sum: 12 + 6 + 3 + 1.5 + ...

7

Find infinite sum: 9 + 3 + 1 + 1/3 + ...

8

Evaluate: Σ (i=1 to 6) 3i

9

Evaluate: Σ (i=1 to 5) (2i + 1)

10

Evaluate: Σ (i=1 to 4) i^2

11

Savings problem: $50 first week, increase by $10 weekly for 12 weeks. Total?

12

Bouncing ball: drops from 20 m, bounces to 0.6 height each time. Total distance?

13

Bouncing ball: drops from 10 m, bounces to half each time. Total distance?

14

Find sum: 1 + 1/2 + 1/4 + 1/8 + ... (infinite)

15

Compare arithmetic vs geometric: which grows faster — 2, 4, 6, 8 or 2, 4, 8, 16?

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. A sequence is to a series as:

- **A.** Sum is to list
- **B.** List is to sum
- **C.** Same thing
- **D.** Opposite

2. Arithmetic series formula:

- **A.** $(n/2)(a_1 + a_n)$
- **B.** $a_1(1 - r^n)/(1 - r)$
- **C.** $a_1/(1 - r)$
- **D.** $a_1 + d$

3. Geometric series formula:

- **A.** $(n/2)(a_1 + a_n)$
- **B.** $a_1(1 - r^n)/(1 - r)$
- **C.** $a_1/(1 - r)$
- **D.** $a_1 \cdot r^n$

4. Infinite geometric series converges when:

- **A.** $r > 1$
- **B.** $r = 1$
- **C.** $|r| < 1$
- **D.** $|r| > 1$

5. Σ (i=1 to 4) i =

- **A.** 4
- **B.** 6
- **C.** 10
- **D.** 24

6. Infinite sum: 8 + 4 + 2 + 1 + ...

- **A.** 8
- **B.** 16
- **C.** 32
- **D.** Diverges

7. Arithmetic series of 4, 7, 10, ... for 10 terms:

- **A.** 175
- **B.** 155
- **C.** 185

- **D.** 165

8. Σ (i=1 to 3) i^2 =

- **A.** 6
- **B.** 9
- **C.** 14
- **D.** 36

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can distinguish between a sequence and a series.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can compute arithmetic and geometric series using formulas.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can determine when an infinite geometric series converges.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can compute the sum of a convergent infinite series.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can read and write sums using sigma notation.
Confident ☐ Mostly ☐ Need Review ☐

Unit 9 Answer Key

Compare your answers below. If you missed a question, review the corresponding lesson section.

Week 32: Arithmetic Sequences

Q#	Answer	Explanation
1	B	$10 - 7 = 3$.
2	B	$a_n = a_1 + (n - 1)d$.
3	C	$a_5 = 8 + 4(2) = 16$.
4	B	$a_1=6$, $d=3$. $a_8 = 6 + 7(3) = 27$.
5	B	$S_n = (n/2)(a_1 + a_n)$.
6	A	$a_5 = 9$. $S_5 = (5/2)(1+9) = 25$.
7	B	Both the first term and the rule (in terms of previous term).
8	B	Constant difference = linear function pattern.

Week 33: Geometric Sequences

Q#	Answer	Explanation
1	B	$12 \div 4 = 3$.
2	B	$a_n = a_1 \cdot r^{n-1}$.
3	C	$a_3 = 10 \cdot 3^2 = 10 \times 9 = 90$.
4	A	$a_1=3$, $r=3$. $a_5 = 3 \cdot 3^4 = 243$.
5	C	$a_1=2$, $r=2$. $a_8 = 2 \cdot 2^7 = 256$.
6	B	Multiplying by a number < 1 each time → decay.
7	C	$a_5 = 100 \cdot 2^4 = 1600$. Wait actually $100 \times 2^5 = 3200$ if "after 5 hours" means 5 doublings.
8	C	$a_4 = 50 \cdot 3^3 = 50 \times 27 = 1350$.

Week 34: Series and Applications

Q#	Answer	Explanation
1	**B**	A sequence is a list; a series adds the list together.
2	**A**	$S_n = (n/2)(a_1 + a_n)$.
3	**B**	$S_n = a_1(1 - r^n)/(1 - r)$ for finite.
4	**C**	Convergence requires $\lvert r \rvert < 1$.
5	**C**	1 + 2 + 3 + 4 = 10.
6	**B**	r = 1/2, S = 8/(1 − 1/2) = 16.
7	**A**	$a_1=4$, d=3. $a_{10}=31$. S = (10/2)(4+31) = 175.
8	**C**	$1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$.

UNIT 10

Probability & Final Review

WEEKS COVERED

Week 35: Probability Concepts
Week 36: Final Review & Exam Preparation

Unit 10 · Week 35

Probability Concepts

Measure the likelihood of events — from a coin flip to weather predictions — using the elegant language of probability.

LESSON OVERVIEW

Welcome to Unit 10. This week we explore probability — the mathematics of chance and uncertainty.

Probability measures how likely an event is to occur. The value always falls between 0 (impossible) and 1 (certain).

For a fair experiment, probability equals favorable outcomes divided by total outcomes.

For example, rolling a 4 on a fair die. There is 1 favorable outcome (the 4 itself) and 6 total outcomes. So P of 4 equals 1 over 6.

There are two types of probability. Theoretical probability is based on counting possible outcomes. Experimental probability is based on actual trials.

The complement rule says: P of NOT A equals 1 minus P of A. If P of rain equals 0.3, then P of no rain equals 0.7.

For independent events — events that do not affect each other — we multiply probabilities. P of A and B equals P of A times P of B.

For example, rolling a 3 AND flipping heads. P equals (1/6) times (1/2) equals 1/12.

For dependent events — where one affects the other — we adjust probabilities. Drawing two red balls from a bag without replacement is dependent.

Conditional probability is the probability of A given B. The formula: P of A given B equals P of A and B divided by P of B.

Probability is used in weather forecasting, finance, medicine, sports, and quality control.

By the end of today, you will compute probabilities of simple, compound, independent, and dependent events.

KEY CONCEPTS & DID YOU KNOW

★ **KEY FORMULA**

P(A) = favorable/total · P(A and B) = P(A)·P(B) if

`independent`

💡 **DID YOU KNOW?**

Probability theory was developed in the 1650s by French mathematicians Pascal and Fermat, who were trying to solve gambling problems for a nobleman. Their letters founded an entire branch of mathematics.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1 Probability of rolling a 4

Step 1	Favorable: 1 outcome (the 4)
Step 2	Total: 6 outcomes

ANSWER P = 1/6

EXAMPLE 2 Sample space for flipping a coin twice

Step 1	List: HH, HT, TH, TT
Step 2	Total: 4 outcomes

ANSWER 4 outcomes

EXAMPLE 3 Complement: P(rain) = 0.3, find P(no rain)

Step 1	P(not A) = 1 − P(A)
Step 2	= 1 − 0.3

ANSWER P(no rain) = 0.7

EXAMPLE 4 P(roll a 3 AND flip heads)

Step 1	P(3) = 1/6, P(H) = 1/2
Step 2	Independent — multiply
Step 3	(1/6)(1/2) = 1/12

ANSWER **1/12**

EXAMPLE 5 **P(both red): 5 red, 3 blue, draw 2 no replace**

Step 1	P(first red) = 5/8
Step 2	P(second red | first red) = 4/7
Step 3	Multiply: 5/8 · 4/7 = 20/56

ANSWER **5/14**

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 P(rolling an even number on a die)

2 P(rolling a number greater than 4 on a die)

3 P(rolling a 5 on a die)

4 List sample space for flipping three coins.

5 Find complement if P(A) = 0.65

6

Find complement if P(rain) = 0.4

7

P(drawing a heart from a deck)

8

P(drawing a king from a deck)

9 P(drawing a red card from a deck)

10 P(roll a 4 AND flip heads)

11 P(flipping HHH on three coins)

12 Independent: P(roll 6) × P(flip tails)

13 Dependent: bag has 4 red, 6 blue. Draw two without replacement. P(both red)?

14 Conditional: P(king | face card)

15 A die is rolled twice. P(sum = 7)?

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. Probability of any event ranges from:

- **A.** −1 to 1
- **B.** 0 to 1
- **C.** 0 to 100
- **D.** 0 to infinity

2. P(rolling an even number on a die):

- **A.** 1/2
- **B.** 1/3
- **C.** 1/6
- **D.** 2/6

3. P(flipping 3 heads in a row):

- **A.** 1/2
- **B.** 1/4
- **C.** 1/8
- **D.** 3/8

4. If P(A) = 0.65, P(not A) =

- **A.** 0.35
- **B.** 0.65
- **C.** 0.5
- **D.** 1.65

5. P(drawing a heart from a standard deck):

- **A.** 1/4
- **B.** 1/13
- **C.** 1/52
- **D.** 13/52

6. Two events that do not affect each other are:

- **A.** Dependent
- **B.** Independent
- **C.** Complementary
- **D.** Disjoint

7. Conditional probability P(A|B) =

- **A.** P(A) + P(B)
- **B.** P(A) · P(B)
- **C.** P(A and B)/P(B)

- **D.** P(A)/P(B)

8. Sample space for flipping 3 coins has how many outcomes?

- **A.** 3
- **B.** 6
- **C.** 8
- **D.** 9

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I can calculate simple probabilities as favorable over total outcomes.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can list a sample space for an experiment.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can apply the complement rule.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can compute probabilities of independent events using multiplication.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can compute conditional probabilities.
Confident ☐ Mostly ☐ Need Review ☐

Unit 10 · Week 36

Final Review & Exam Preparation

Bring it all together — review the entire Algebra 2 journey from real numbers to probability, ready for the final exam.

LESSON OVERVIEW

Welcome to Week 36 — the final week of Algebra 2. Today we look back at everything we have learned and prepare for the comprehensive exam.

We started with the foundation: real numbers, linear equations, and the basics of algebra. These tools support every later topic.

In Unit 2, we met functions — the language for describing how one quantity depends on another. Function notation, domain, range, and types.

Unit 3 covered polynomials: adding, subtracting, multiplying, dividing, factoring, and advanced factoring including cubes and substitution.

Unit 4 introduced rational expressions — fractions of polynomials — along with how to simplify, multiply, divide, solve equations, and apply them to real life.

Unit 5 was about radicals — square roots and how to simplify, operate on, and solve equations involving them.

In Unit 6, we tackled quadratic functions: graphing parabolas, factoring, completing the square, the quadratic formula, and applications.

Unit 7 brought exponential and logarithmic functions — modeling explosive growth, decay, and using logarithms to unravel exponentials.

Unit 8 took us into systems of equations — two and three variables, with mixture, rate, and supply-demand applications.

Unit 9 covered sequences and series, both arithmetic and geometric, and the magic of convergent infinite series.

Unit 10 introduced probability — measuring uncertainty with elegance and precision.

To prepare for the exam: practice problems from every unit, focus on understanding rather than memorization, identify your weak areas, and manage time during the test.

Algebra 2 is a doorway. Beyond it lie precalculus, calculus, statistics — and countless applications in science, technology, and life.

KEY CONCEPTS & DID YOU KNOW

★ KEY FORMULA

Master what you know · Practice what you don't · Trust the journey

💡 DID YOU KNOW?

Algebra is one of humanity's greatest intellectual inventions. The word "algebra" comes from the Arabic "al-jabr," from a 9th-century book by Persian mathematician Al-Khwarizmi. The word means "reunion of broken parts" — exactly what algebra does when solving equations.

WORKED EXAMPLES

Study these examples carefully. Cover the steps with a piece of paper and try to solve each problem on your own before reading the solution.

EXAMPLE 1 Linear: solve $3x + 5 = 20$

Step 1	Subtract 5: $3x = 15$
Step 2	Divide by 3: $x = 5$

ANSWER $x = 5$

EXAMPLE 2 Quadratic: solve $x^2 - 5x + 6 = 0$

Step 1	Factor: $(x - 2)(x - 3) = 0$
Step 2	Set each = 0: $x = 2$ or $x = 3$

ANSWER $x = 2, 3$

EXAMPLE 3 Exponential: solve $2^x = 16$

Step 1	Rewrite: $2^x = 2^4$
Step 2	Match exponents: $x = 4$

ANSWER $x = 4$

EXAMPLE 4 System: $x + y = 10$, $x - y = 2$

Step 1	Add equations: $2x = 12$
Step 2	$x = 6$
Step 3	Substitute: $y = 4$

ANSWER **(6, 4)**

EXAMPLE 5 **Arithmetic: 10th term of 2, 5, 8, 11**

Step 1	$a_1 = 2, d = 3$
Step 2	$a_{10} = 2 + 9(3) = 29$

ANSWER **29**

PRACTICE PROBLEMS

Show all work in the space provided. Use additional paper if needed.

1 Solve: $4x - 7 = 21$

2 Solve: $x^2 - 9 = 0$

3 Solve: 3^x = 81

4 Solve: $x + y = 12$, $x - y = 4$

5 Find 12th term: 5, 8, 11

6

Find sum: 2 + 4 + 6 + ... (10 terms)

7

P(drawing a red card from a deck)

8

Factor: $x^2 + 5x + 6$

9 Simplify: $(x^2 - 4)/(x - 2)$

10 Solve: $\log_2(x) = 4$

11 Find domain of $f(x) = 1/(x - 3)$

12 Solve by quadratic formula: $x^2 - 5x + 3 = 0$

13 Solve system: $2x + y = 7$, $x - y = 2$

14 Find common ratio: 4, 12, 36, 108

15 P(rolling 4 on a die AND flipping heads)

SELF-CHECK QUIZ

Answer each question without referring to your notes. Circle the letter of your answer. Check your answers in the unit answer key at the end of this unit.

1. Solve $4x - 7 = 21$

- **A.** $x = 7$
- **B.** $x = 14$
- **C.** $x = 5$
- **D.** $x = 4$

2. Solve $x^2 - 9 = 0$

- **A.** $x = 9$
- **B.** $x = \pm 3$
- **C.** $x = 3$
- **D.** $x = \pm 9$

3. Solve $3^x = 81$

- **A.** $x = 3$
- **B.** $x = 4$
- **C.** $x = 27$
- **D.** $x = 81$

4. For $x + y = 12$, $x - y = 4$, $x =$

- **A.** 4
- **B.** 6
- **C.** 8
- **D.** 12

5. 12th term of 5, 8, 11, ...

- **A.** 33
- **B.** 36
- **C.** 38
- **D.** 41

6. Sum: 2 + 4 + 6 + ... (10 terms)

- **A.** 100
- **B.** 110
- **C.** 55
- **D.** 220

7. P(drawing a red card from a standard deck):

- **A.** 1/4
- **B.** 1/2
- **C.** 1/13

- **D.** 13/26

8. The Pythagorean theorem solves for:

- **A.** Quadratic roots
- **B.** Triangle side lengths
- **C.** Probability
- **D.** Logarithms

SELF-ASSESSMENT

Rate yourself on each skill below. Circle one: Confident / Mostly / Need Review

✓ I have reviewed all 10 units of Algebra 2.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can solve a variety of equations: linear, quadratic, exponential, logarithmic, rational, and radical.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can analyze functions: identify type, find domain/range, and evaluate.
Confident ☐ Mostly ☐ Need Review ☐

✓ I can solve systems and work with sequences and series.
Confident ☐ Mostly ☐ Need Review ☐

✓ I am ready for the Algebra 2 final exam.
Confident ☐ Mostly ☐ Need Review ☐

Unit 10 Answer Key

Compare your answers below. If you missed a question, review the corresponding lesson section.

Week 35: Probability Concepts

Q#	Answer	Explanation
1	**B**	Probabilities are between 0 (impossible) and 1 (certain).
2	**A**	Even outcomes: 2, 4, 6 (three of six) = 1/2.
3	**C**	Independent: $(1/2)^3 = 1/8$.
4	**A**	Complement: 1 − 0.65 = 0.35.
5	**A**	13 hearts in 52 cards = 13/52 = 1/4.
6	**B**	Independent events: outcome of one does not change the other.
7	**C**	$P(A\|B) = P(A \text{ and } B)/P(B)$.
8	**C**	2 × 2 × 2 = 8 outcomes.

Week 36: Final Review & Exam Preparation

Q#	Answer	Explanation
1	**A**	$4x = 28$, $x = 7$.
2	**B**	$x^2 = 9$, $x = \pm 3$.
3	**B**	$81 = 3^4$, so $x = 4$.
4	**C**	Add: $2x = 16$, $x = 8$.
5	**C**	$a_1=5$, $d=3$. $a_{12} = 5 + 11(3) = 38$.
6	**B**	$a_1=2$, $d=2$, $a_{10}=20$. $S = (10/2)(2+20) = 110$.
7	**B**	26 red cards out of 52 = 1/2.
8	**B**	$a^2 + b^2 = c^2$ for right triangles.

🎓 Congratulations!

You have completed Grade 11 Algebra 2.

"Mathematics is the music of reason." — James Joseph Sylvester

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