BEMANDASTEM ACADEMY

PRECALCULUS · GRADE 12

Companion Workbook

A complete print-ready companion to the BemandaSTEM Grade 12 Precalculus offline learning app

9	36	545+
UNITS	WEEKS	EXERCISES

Student Name: ________________________________

Teacher: ________________________________

Class / Period: ________________________________

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HOW TO USE THIS WORKBOOK

This companion workbook accompanies the BemandaSTEM Grade 12 Precalculus offline app. The two are designed to work together: the app teaches each concept interactively with narrated lessons, randomised practice, and self-checking quizzes, while this workbook provides print-ready exercises, work space, summaries, and answer keys for offline study, homework, and assessment.

Structure of each week

- **Learning Objectives** — eight clear goals for the week.
- **Key Concepts** — the core ideas distilled into concise prose, matching the app's five teaching panels.
- **Worked Examples** — step-by-step worked problems with explanations.
- **Practice Exercises** — fifteen exercises with work space for the student.
- **Quiz Practice** — eight multiple-choice questions in the format of the app's quiz.
- **Did You Know?** — a curated mathematical, historical, or scientific connection.
- **Cheat Sheet Summary** — quick-reference formulas and key facts.

Answer keys

Every unit ends with a complete answer key covering all practice exercises and quiz questions for that unit. Teachers may remove these pages before distributing if they wish students to complete the workbook independently.

Pacing

The workbook follows the app's 36-week structure, one week per workbook section. A typical pacing assumes 4 to 5 hours of work per week, split between app interaction, workbook exercises, and assessment. Each week is designed to be self-contained — students who fall behind can complete weeks at their own pace without losing continuity.

Symbols used

Mathematical notation in this workbook follows standard secondary school conventions: superscripts for exponents (x^2), Greek letters for angles (θ, π), the limit symbol (lim), and standard inequality symbols ($\leq$, $\geq$, $\neq$). Where the app uses interactive widgets, the workbook substitutes static diagrams and worked notation.

UNIT 1
Review of Functions and Algebra

UNIT OVERVIEW

Meet the family of functions — linear, quadratic, polynomial, rational, radical, exponential, logarithmic, absolute value, and piecewise. Learn transformations, inverses, and how to combine these ideas for real-world problems.

Weeks in this unit:

Week 1 — *Function Types and Properties*
Week 2 — *Transformations of Functions*
Week 3 — *Inverses of Functions*
Week 4 — *Applications and Review*

UNIT 1 · REVIEW OF FUNCTIONS AND ALGEBRA

WEEK 1

Function Types and Properties

Meet the family of functions — linear, quadratic, polynomial, rational, radical, exponential, logarithmic, absolute value, and piecewise. Learn how to evaluate them, find their domain and range, and read their behaviour from a graph.

Learning Objectives

By the end of this week, you will be able to:

1. Define a function and use function notation correctly.
2. Classify different types of functions and recognise their shape.
3. Determine the domain and range of a function.
4. Evaluate a function at any input value.
5. Identify increasing and decreasing intervals.
6. Recognise even and odd functions from algebra and graph.
7. Interpret graphs of common functions.
8. Model real-world relationships with the right function type.

Key Concepts

1. What is a function?

A function is a rule that takes an input and returns exactly one output. We write it as f of x. For example, f of x equals two x plus three means: take the input, multiply by two, then add three.

2. The function family

There are nine common families of functions you must recognise. Each has a signature shape and a typical equation.

3. Domain and range

The domain is the set of allowed inputs. The range is the set of possible outputs.

4. Even, odd, and symmetry

An even function is symmetric about the y-axis: f of minus x equals f of x. An odd function is symmetric about the origin: f of minus x equals minus f of x.

5. Worked examples

Let us work through five examples together.

Worked Examples

Example 1. Evaluate $f(x) = 3x^2 - 2x + 1$ at $x = 2$.

Solution: $f(2) = 12 - 4 + 1 = 9$.

Example 2. Evaluate $f(x) = 5x - 4$ at $x = -3$.

Solution: $f(-3) = -15 - 4 = -19$.

Example 3. Find the domain of $f(x) = 1/(x - 7)$.

Solution: All real numbers except $x = 7$.

UNIT 1 · REVIEW OF FUNCTIONS AND ALGEBRA

Week 1 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Evaluate $f(x) = 3x^2 - 2x + 1$ at $x = 2$.

2. Evaluate $f(x) = 5x - 4$ at $x = -3$.

3. Find the domain of $f(x) = 1/(x - 7)$.

4. Find the domain of $f(x) = \sqrt{(2x - 6)}$.

5. Classify $f(x) = 4x^3 - x$: even, odd, or neither?

6. Classify $f(x) = x^4 + 3x^2 + 1$: even, odd, or neither?

7. Classify $f(x) = x^2 + x$: even, odd, or neither?

8. What type of function is $f(x) = \log_2(x)$?

9. What type of function is $f(x) = (x^2 + 1)/(x - 2)$?

10. Is $f(x) = -2x + 5$ increasing or decreasing?

11. Find $f(0)$ for $f(x) = 2 \cdot 3^x$.

12. For $f(x) = \{\ x+5 \text{ if } x<2\ ;\ 3x \text{ if } x \geq 2\ \}$, find $f(1)$ and $f(4)$.

13. State the domain and range of $f(x) = |x|$.

14. A model $T(x) = 4x + 12$ gives temperature after x hours. Find $T(6)$.

15. For $T(x) = 4x + 12$, what value of x produces $T = 60$?

UNIT 1 · REVIEW OF FUNCTIONS AND ALGEBRA

Week 1 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. Which of the following is the correct definition of a function?

(A) A graph that always passes through the origin.
(B) Any equation involving x and y.
(C) A relation where each input has exactly one output.
(D) A relation where each output has exactly one input.
Your answer: ____________

2. What is f(4) if $f(x) = x^2 - 2x + 1$?

(A) 9
(B) 17
(C) 7
(D) 25
Your answer: ____________

3. What is the domain of $f(x) = \sqrt{(x - 5)}$?

(A) $x \leq 5$
(B) All real numbers
(C) $x > 0$
(D) $x \geq 5$
Your answer: ____________

4. Which function is even?

(A) $f(x) = x^3 + x$
(B) $f(x) = x + 1$
(C) $f(x) = x^4 + 2$
(D) $f(x) = x^3$
Your answer: ____________

5. What value must be excluded from the domain of $f(x) = 1/(x + 2)$?

(A) $x = -2$
(B) $x = 2$
(C) $x = 1$
(D) $x = 0$
Your answer: ____________

6. Which describes the graph of $f(x) = |x|$?

(A) Smooth growing curve.
(B) V-shaped, vertex at origin, always non-negative.
(C) Straight line through origin.

(D) U-shaped parabola.

Your answer: ____________

7. A function $f(x) = 2 \cdot 3^x$ is what type of function?

(A) Quadratic
(B) Radical
(C) Linear
(D) Exponential

Your answer: ____________

8. For the piecewise function $f(x) = \{\ 2x \text{ if } x \geq 0\ ;\ x - 1 \text{ if } x < 0\ \}$, what is $f(-3)$?

(A) −2
(B) −6
(C) 3
(D) −4

Your answer: ____________

DID YOU KNOW?

The word "function" was first used in mathematics by Gottfried Leibniz in 1673 — over 350 years ago. The notation f(x) we still use today was introduced by Leonhard Euler in 1734, and it remains the most powerful single notation in all of algebra.

UNIT 1 · REVIEW OF FUNCTIONS AND ALGEBRA

WEEK 2

Transformations of Functions

Functions can be shifted, reflected, stretched, and compressed. Master the general form $g(x) = a \cdot f(b(x - h)) + k$ and you can move any graph to any place on the plane.

Learning Objectives

By the end of this week, you will be able to:

1. Identify common function transformations.
2. Apply vertical and horizontal shifts correctly.
3. Perform reflections over the x-axis and y-axis.
4. Apply stretches and compressions to functions.
5. Combine multiple transformations in the correct order.
6. Interpret transformed graphs.
7. Compare parent and transformed functions.
8. Apply transformations to real-world models.

Key Concepts

1. The big idea

A transformation is a change applied to a function that moves, flips, or reshapes its graph without changing what kind of function it is.

2. Vertical and horizontal shifts

Adding outside the function shifts up or down. Subtracting inside the parentheses shifts right; adding inside shifts left.

3. Reflections, stretches, compressions

A negative sign reflects. A constant multiplier stretches or compresses.

4. Order of transformations

When combining transformations, follow this order: horizontal shift, then horizontal stretch, then reflections, then vertical stretch, then vertical shift.

5. Worked examples

Five worked examples follow.

Worked Examples

Example 1. Describe the transformation: $f(x) = x^2 \rightarrow f(x) + 5$.

Solution: Shift up 5 units.

Example 2. Describe the transformation: $f(x) = x^2 \to f(x - 4)$.

Solution: Shift right 4 units.

Example 3. Describe the transformation: $f(x) = x^2 \to -f(x)$.

Solution: Reflect over the x-axis.

UNIT 1 · REVIEW OF FUNCTIONS AND ALGEBRA

Week 2 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Describe the transformation: $f(x) = x^2 \to f(x) + 5$.

2. Describe the transformation: $f(x) = x^2 \to f(x - 4)$.

3. Describe the transformation: $f(x) = x^2 \to -f(x)$.

4. Describe the transformation: $f(x) = x^2 \to f(-x)$.

5. Describe the transformation: $f(x) = x^2 \to 3f(x)$.

6. Where is the vertex of $g(x) = (x - 5)^2 + 3$?

7. Where is the vertex of $g(x) = -(x + 2)^2 - 1$?

8. Write the equation for $f(x) = \sqrt{x}$ shifted right 3 and down 2.

9. Write the equation for $f(x) = |x|$ shifted left 4 and up 6.

10. What parent function is hidden in $g(x) = 2(x - 1)^3 + 4$?

11. In $g(x) = 0.5 \cdot f(x)$, what happens to the graph?

12. In $g(x) = f(2x)$, what happens to the graph?

13. A model $T(x) = (x - 5)^2 + 10$ represents temperature. Describe the transformation from x^2.

14. Sketch the steps to go from x^2 to $-(x + 1)^2 + 6$.

15. List the transformations in $g(x) = 2(x - 3)^2 - 1$ from the parent.

UNIT 1 · REVIEW OF FUNCTIONS AND ALGEBRA

Week 2 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. What does the +3 in $g(x) = f(x) + 3$ do?

(A) Shifts the graph up 3 units.
(B) Stretches by 3.
(C) Shifts the graph right 3 units.
(D) Reflects the graph.
Your answer: ____________

2. What does $(x - 5)$ inside the function do?

(A) Shifts the graph right 5.
(B) Compresses vertically.
(C) Shifts the graph left 5.
(D) Reflects over y-axis.
Your answer: ____________

3. $g(x) = -f(x)$ reflects the graph over which axis?

(A) line $y = x$
(B) origin
(C) y-axis
(D) x-axis
Your answer: ____________

4. Which transformation does $g(x) = 3f(x)$ describe?

(A) Shift up 3
(B) Horizontal stretch by 3
(C) Vertical compression
(D) Vertical stretch by 3
Your answer: ____________

5. Where is the vertex of $g(x) = (x + 2)^2 - 5$?

(A) $(-2, -5)$
(B) $(2, 5)$
(C) $(2, -5)$
(D) $(-2, 5)$
Your answer: ____________

6. Write an equation for $y = x^2$ shifted right 4 and up 7.

(A) $y = (x - 4)^2 + 7$
(B) $y = (x + 4)^2 + 7$
(C) $y = x^2 - 4 + 7$

(D) $y = (x - 4)^2 - 7$

Your answer: ____________

7. Which is the correct order for applying combined transformations?

(A) Reflections first, always

(B) Horizontal shift → horizontal stretch → reflections → vertical stretch → vertical shift

(C) Vertical shift first, then everything else

(D) Order does not matter

Your answer: ____________

8. The parent function of $g(x) = \sqrt{(x - 2)} + 3$ is …

(A) $f(x) = |x|$

(B) $f(x) = \sqrt{x}$

(C) $f(x) = 3^x$

(D) $f(x) = x^2$

Your answer: ____________

DID YOU KNOW?

Computer graphics, video games, and animation rely on exactly these transformations — every time a character moves on screen, the engine is applying horizontal shifts, vertical shifts, and scaling to thousands of points per second. The maths you learn this week powers every Pixar film.

UNIT 1 · REVIEW OF FUNCTIONS AND ALGEBRA

WEEK 3

Inverses of Functions

Every one-to-one function has a twin that undoes it: the inverse. Learn the horizontal line test, the swap-and-solve algorithm, and how to verify inverses with composition.

Learning Objectives

By the end of this week, you will be able to:

1. Define an inverse function.
2. Determine whether a function has an inverse.
3. Apply the horizontal line test.
4. Find inverses of simple algebraic functions.
5. Verify inverses using composition.
6. Identify domain and range restrictions for inverses.
7. Graph inverse relationships as reflections over $y = x$.
8. Apply inverse functions to real-world problems.

Key Concepts

1. What is an inverse function?

An inverse function undoes the original. If f takes input x to output y, the inverse takes y back to x.

2. One-to-one and the horizontal line test

A function has an inverse only if it is one-to-one, meaning each output corresponds to exactly one input. We test this with the horizontal line test.

3. Find an inverse — four steps

To find an inverse algebraically: replace f of x with y, swap x and y, solve for y, and rewrite as f inverse of x.

4. Verify with composition

Two functions are inverses if f of f inverse of x equals x, and f inverse of f of x equals x.

5. Worked examples

Five inverse examples.

Worked Examples

Example 1. Find the inverse of $f(x) = x + 7$.

Solution: $f^{-1}(x) = x - 7$.

Example 2. Find the inverse of $f(x) = 5x - 2$.

Solution: $f^{-1}(x) = (x + 2)/5$.

Example 3. Find the inverse of $f(x) = (x - 4)/3$.

Solution: $f^{-1}(x) = 3x + 4$.

UNIT 1 · REVIEW OF FUNCTIONS AND ALGEBRA

Week 3 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Find the inverse of $f(x) = x + 7$.

2. Find the inverse of $f(x) = 5x - 2$.

3. Find the inverse of $f(x) = (x - 4)/3$.

4. Does $f(x) = x^3$ have an inverse on all reals?

5. Does $f(x) = x^2$ have an inverse on all reals?

6. Find the inverse of $f(x) = (x - 1)^2$, $x \geq 1$.

7. Verify $f(x) = 2x + 1$ and $g(x) = (x - 1)/2$ are inverses.

8. If f has domain [0, 10] and range [−2, 8], what is the range of f^{-1}?

9. Apply: d(t) = 4t + 6 models distance. Find the inverse and interpret.

10. Find f^{-1} if $f(x) = \sqrt{(x + 4)}$.

11. Apply: $V(r) = (4/3)\pi r^3$ (sphere volume). Find r in terms of V.

12. Convert 100°F to Celsius using C(F) = (5/9)(F − 32).

13. Why must f(x) = sin(x) be restricted to [−π/2, π/2] for arcsin?

14. Find the inverse of f(x) = (2x + 1)/3.

15. Find the inverse of f(x) = 1/x.

UNIT 1 · REVIEW OF FUNCTIONS AND ALGEBRA

Week 3 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. Which notation represents the inverse of f?

- **(A)** $f(x)^{-1}$
- **(B)** $1/f(x)$
- **(C)** $f^{-1}(x)$
- **(D)** $f^2(x)$

Your answer: ____________

2. A function has an inverse if and only if it is…

- **(A)** quadratic
- **(B)** continuous
- **(C)** one-to-one
- **(D)** increasing

Your answer: ____________

3. Find the inverse of $f(x) = 2x + 6$.

- **(A)** $f^{-1}(x) = 2x - 6$
- **(B)** $f^{-1}(x) = (x - 6)/2$
- **(C)** $f^{-1}(x) = (x + 6)/2$
- **(D)** $f^{-1}(x) = x/2 - 6$

Your answer: ____________

4. Verify: are $f(x) = 2x + 1$ and $g(x) = (x - 1)/2$ inverses?

- **(A)** No — they differ at $x = 0$.
- **(B)** Cannot tell.
- **(C)** Only when $x > 0$.
- **(D)** Yes — composition gives x both ways.

Your answer: ____________

5. A function and its inverse are reflections across which line?

- **(A)** y-axis
- **(B)** $y = 0$
- **(C)** $y = x$
- **(D)** x-axis

Your answer: ____________

6. Why does $f(x) = x^2$ need a domain restriction before inverting?

- **(A)** It is too complicated.
- **(B)** It is undefined.
- **(C)** It is not one-to-one without restriction.

(D) It does not.

Your answer: ____________

7. The inverse of F(C) = (9/5)C + 32 is…

(A) C(F) = (5/9)(F − 32)
(B) C(F) = F/9 − 32
(C) C(F) = (9/5)F + 32
(D) C(F) = (5/9)F − 32

Your answer: ____________

DID YOU KNOW?

Every cryptographic system relies on inverse functions. When you log in to a website, your password is run through a one-way function — chosen specifically because it has no easy inverse. The strength of modern encryption is the difficulty of inverting these functions, even with a supercomputer.

UNIT 1 · REVIEW OF FUNCTIONS AND ALGEBRA

WEEK 4

Applications and Review

Bring it all together. Combine function types, transformations, and inverses to tackle multi-step problems and real-world models. This is your unit review and assessment prep.

Learning Objectives

By the end of this week, you will be able to:

1. Review and apply key function concepts.
2. Analyse transformations and inverses together.
3. Solve multi-step function problems.
4. Interpret graphs and equations accurately.
5. Determine domain and range in complex problems.
6. Apply functions to real-world situations.
7. Connect multiple function concepts in one problem.
8. Demonstrate readiness for assessment.

Key Concepts

1. The big picture

In this week we bring together everything from Weeks 1, 2, and 3. Most real-world precalculus problems use several concepts at once.

2. Problem-solving strategy

Follow these six steps for any precalculus function problem.

3. Reading graphs fluently

Graphs tell you everything: where the function lives, how it moves, whether it has an inverse, and how its parts connect.

4. Worked examples

Five integration examples.

Worked Examples

Example 1. Evaluate $f(x) = 2x + 5$ at $x = -4$.

Solution: $f(-4) = -8 + 5 = -3$.

Example 2. Evaluate $f(x) = x^2 - 3$ at $x = 4$.

Solution: $f(4) = 16 - 3 = 13$.

Example 3. Describe the transformation from x^2 to $(x - 1)^2 + 4$.

Solution: Right 1, up 4. Vertex (1, 4).

UNIT 1 · REVIEW OF FUNCTIONS AND ALGEBRA

Week 4 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Evaluate $f(x) = 2x + 5$ at $x = -4$.

2. Evaluate $f(x) = x^2 - 3$ at $x = 4$.

3. Describe the transformation from x^2 to $(x - 1)^2 + 4$.

4. Describe the transformation from x^2 to $-(x + 2)^2 - 1$.

5. Find the inverse of $f(x) = 4x - 7$.

6. Find the inverse of $f(x) = (x - 3)/2$.

7. State domain and range of $f(x) = 1/(x + 2)$.

8. State domain and range of $f(x) = \sqrt{x}$.

9. A distance model is $d(t) = 5t + 2$. Find $d(6)$.

10. For $d(t) = 5t + 2$, find the inverse and interpret.

11. Determine whether $f(x) = 3x^5 - x$ is even, odd, or neither.

12. Determine whether $f(x) = x^2 + |x|$ is even, odd, or neither.

13. Identify the parent function of $g(x) = 2|x - 3| + 5$.

14. Identify the parent function of $g(x) = \sqrt{(x + 7)} - 2$.

15. If $T(x) = 4x + 10$ models temperature, when does $T = 50$?

UNIT 1 · REVIEW OF FUNCTIONS AND ALGEBRA

Week 4 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. Describe the transformation from x^2 to $(x + 3)^2 - 4$.

(A) Left 3, up 4
(B) Right 3, down 4
(C) Right 3, up 4
(D) Left 3, down 4
Your answer: ____________

2. Find $f(-1)$ for $f(x) = x^3 + 2x$.

(A) 3
(B) −3
(C) 1
(D) −1
Your answer: ____________

3. Find the inverse of $f(x) = 4x - 7$.

(A) $f^{-1}(x) = (x + 7)/4$
(B) $f^{-1}(x) = x/4 + 7$
(C) $f^{-1}(x) = (x - 7)/4$
(D) $f^{-1}(x) = 4x + 7$
Your answer: ____________

4. State the domain of $f(x) = 1/(x + 2)$.

(A) All real numbers except $x = -2$
(B) $x > 0$
(C) $x \geq -2$
(D) All real numbers
Your answer: ____________

5. State the domain of $f(x) = \sqrt{x}$.

(A) $x > 0$
(B) $x \geq 0$
(C) All real numbers
(D) $x \leq 0$
Your answer: ____________

6. A function $d(t) = 5t + 2$ models distance. Find the inverse $t(d)$.

(A) $t(d) = (d + 2)/5$
(B) $t(d) = 5d + 2$
(C) $t(d) = 5(d - 2)$

(D) $t(d) = (d - 2)/5$

Your answer: ____________

7. Which is even? Which is odd?

(A) Both odd.

(B) Both even.

(C) x^4 is odd; x^3 is even.

(D) x^4 is even; x^3 is odd.

Your answer: ____________

8. If $f(x) = \sqrt{(x - 4)}$, find $f(13)$.

(A) 3

(B) 9

(C) 17

(D) -3

Your answer: ____________

DID YOU KNOW?

Real-world functions almost never come neatly labelled. Engineers spend years learning to look at messy data — a noisy temperature curve, a stock price, a growth rate — and decide which function family models it best. That is the skill this unit is preparing you for: recognising a function in the wild.

ANSWER KEY

Unit 1 · Review of Functions and Algebra

This answer key covers every practice exercise and quiz question from Unit 1. For full step-by-step solutions to randomised practice generators (separate from the worksheet exercises printed here), refer to the BemandaSTEM Precalculus app.

ANSWER KEY

Week 1 — Function Types and Properties

Practice Exercises

1. **Q.** Evaluate $f(x) = 3x^2 - 2x + 1$ at $x = 2$.

   A. $f(2) = 12 - 4 + 1 = 9$.

2. **Q.** Evaluate $f(x) = 5x - 4$ at $x = -3$.

   A. $f(-3) = -15 - 4 = -19$.

3. **Q.** Find the domain of $f(x) = 1/(x - 7)$.

   A. All real numbers except $x = 7$.

4. **Q.** Find the domain of $f(x) = \sqrt{(2x - 6)}$.

   A. $2x - 6 \geq 0$, so $x \geq 3$.

5. **Q.** Classify $f(x) = 4x^3 - x$: even, odd, or neither?

   A. Odd (all odd powers and no constant).

6. **Q.** Classify $f(x) = x^4 + 3x^2 + 1$: even, odd, or neither?

   A. Even (all even powers).

7. **Q.** Classify $f(x) = x^2 + x$: even, odd, or neither?

   A. Neither.

8. **Q.** What type of function is $f(x) = \log_2(x)$?

   A. Logarithmic function.

9. **Q.** What type of function is $f(x) = (x^2 + 1)/(x - 2)$?

   A. Rational function.

10. **Q.** Is $f(x) = -2x + 5$ increasing or decreasing?

    A. Decreasing (slope is negative).

11. **Q.** Find $f(0)$ for $f(x) = 2 \cdot 3^x$.

    A. $f(0) = 2 \cdot 1 = 2$.

12. **Q.** For $f(x) = \{ x+5 \text{ if } x<2 \text{ ; } 3x \text{ if } x\geq2 \}$, find $f(1)$ and $f(4)$.

    A. $f(1) = 6$; $f(4) = 12$.

13. **Q.** State the domain and range of $f(x) = |x|$.

    A. Domain: all real numbers; Range: $y \geq 0$.

14. **Q.** A model $T(x) = 4x + 12$ gives temperature after x hours. Find $T(6)$.

    A. $T(6) = 24 + 12 = 36$.

15. **Q.** For $T(x) = 4x + 12$, what value of x produces $T = 60$?

    A. $4x + 12 = 60 \Rightarrow x = 12$.

Quiz Answers

1. **Answer: (C) A relation where each input has exactly one output.**

   Reason: A function assigns exactly one output to each input — the vertical-line-test rule.

2. **Answer: (A) 9**

   Reason: $f(4) = 16 - 8 + 1 = 9$.

3. **Answer: (D) $x \geq 5$**

   Reason: Inside the square root must be ≥ 0, so $x - 5 \geq 0$, i.e. $x \geq 5$.

4. **Answer: (C) $f(x) = x^4 + 2$**

   Reason: Only $x^4 + 2$ satisfies $f(-x) = f(x)$. Even functions are symmetric about the y-axis.

5. **Answer: (A) $x = -2$**

   Reason: Set $x + 2 = 0$; the denominator is zero when $x = -2$.

6. **Answer: (B) V-shaped, vertex at origin, always non-negative.**

   Reason: The absolute value function makes a V with vertex (0,0); outputs are never negative.

7. **Answer: (D) Exponential**

   Reason: A constant times a base raised to x is the signature of an exponential function.

8. **Answer: (D) −4**

   Reason: Since $-3 < 0$ use $x - 1$: $-3 - 1 = -4$.

ANSWER KEY

Week 2 — Transformations of Functions

Practice Exercises

1. **Q.** Describe the transformation: $f(x) = x^2 \to f(x) + 5$.
   A. Shift up 5 units.
2. **Q.** Describe the transformation: $f(x) = x^2 \to f(x - 4)$.
   A. Shift right 4 units.
3. **Q.** Describe the transformation: $f(x) = x^2 \to -f(x)$.
   A. Reflect over the x-axis.
4. **Q.** Describe the transformation: $f(x) = x^2 \to f(-x)$.
   A. Reflect over the y-axis.
5. **Q.** Describe the transformation: $f(x) = x^2 \to 3f(x)$.
   A. Vertical stretch by factor 3.
6. **Q.** Where is the vertex of $g(x) = (x - 5)^2 + 3$?
   A. (5, 3).
7. **Q.** Where is the vertex of $g(x) = -(x + 2)^2 - 1$?
   A. (−2, −1); opens downward.
8. **Q.** Write the equation for $f(x) = \sqrt{x}$ shifted right 3 and down 2.
   A. $g(x) = \sqrt{(x - 3)} - 2$.
9. **Q.** Write the equation for $f(x) = |x|$ shifted left 4 and up 6.
   A. $g(x) = |x + 4| + 6$.
10. **Q.** What parent function is hidden in $g(x) = 2(x - 1)^3 + 4$?
    A. $f(x) = x^3$ (cubic).
11. **Q.** In $g(x) = 0.5 \cdot f(x)$, what happens to the graph?
    A. Vertical compression by factor 0.5.
12. **Q.** In $g(x) = f(2x)$, what happens to the graph?
    A. Horizontal compression by factor 2.
13. **Q.** A model $T(x) = (x - 5)^2 + 10$ represents temperature. Describe the transformation from x^2.
    A. Shifted right 5 and up 10.
14. **Q.** Sketch the steps to go from x^2 to $-(x + 1)^2 + 6$.
    A. Left 1 → reflect over x-axis → up 6. Vertex (−1, 6), opens down.
15. **Q.** List the transformations in $g(x) = 2(x - 3)^2 - 1$ from the parent.
    A. Right 3, vertical stretch by 2, down 1.

Quiz Answers

1. **Answer: (A) Shifts the graph up 3 units.**
   Reason: Adding outside the function moves the entire graph vertically — here, up by 3.
2. **Answer: (A) Shifts the graph right 5.**
   Reason: Inside the parentheses the sign flips: x − 5 shifts the graph right 5 units.

3. Answer: (D) x-axis

Reason: A negative in front of f flips the output values — reflection across the x-axis.

4. Answer: (D) Vertical stretch by 3

Reason: A constant > 1 outside multiplies the output, stretching the graph vertically by factor 3.

5. Answer: (A) $(-2, -5)$

Reason: Inside parentheses sign flips: $(x + 2)$ means shift left 2. Then subtract 5 to shift down 5. Vertex $(-2, -5)$.

6. Answer: (A) $y = (x - 4)^2 + 7$

Reason: Right h: $(x - h)$. Up k: $+ k$. So $(x - 4)^2 + 7$.

7. Answer: (B) Horizontal shift → horizontal stretch → reflections → vertical stretch → vertical shift

Reason: Inside-out: handle horizontal effects first, then reflections, then vertical effects.

8. Answer: (B) $f(x) = \sqrt{x}$

Reason: Stripping shifts, the underlying rule is the square root function $f(x) = \sqrt{x}$.

ANSWER KEY

Week 3 — Inverses of Functions

Practice Exercises

1. **Q.** Find the inverse of $f(x) = x + 7$.
   A. $f^{-1}(x) = x - 7$.
2. **Q.** Find the inverse of $f(x) = 5x - 2$.
   A. $f^{-1}(x) = (x + 2)/5$.
3. **Q.** Find the inverse of $f(x) = (x - 4)/3$.
   A. $f^{-1}(x) = 3x + 4$.
4. **Q.** Does $f(x) = x^3$ have an inverse on all reals?
   A. Yes — it is monotonic.
5. **Q.** Does $f(x) = x^2$ have an inverse on all reals?
   A. No — fails horizontal line test.
6. **Q.** Find the inverse of $f(x) = (x - 1)^2$, $x \geq 1$.
   A. $f^{-1}(x) = \sqrt{x} + 1$.
7. **Q.** Verify $f(x) = 2x + 1$ and $g(x) = (x - 1)/2$ are inverses.
   A. $f(g(x)) = 2\cdot(x-1)/2 + 1 = x$. $g(f(x)) = (2x+1-1)/2 = x$. Yes.
8. **Q.** If f has domain [0, 10] and range [−2, 8], what is the range of f^{-1}?
   A. [0, 10].
9. **Q.** Apply: $d(t) = 4t + 6$ models distance. Find the inverse and interpret.
   A. $t = (d - 6)/4$. Given a distance d, this gives the time t to reach it.
10. **Q.** Find f^{-1} if $f(x) = \sqrt{(x + 4)}$.
    A. $f^{-1}(x) = x^2 - 4$, $x \geq 0$.
11. **Q.** Apply: $V(r) = (4/3)\pi r^3$ (sphere volume). Find r in terms of V.
    A. $r = \sqrt[3]{(3V/(4\pi))}$.
12. **Q.** Convert 100°F to Celsius using $C(F) = (5/9)(F - 32)$.
    A. $C(100) = (5/9)\cdot 68 \approx 37.8$°C.
13. **Q.** Why must $f(x) = \sin(x)$ be restricted to $[-\pi/2, \pi/2]$ for arcsin?
    A. To make it one-to-one — otherwise sine repeats and the inverse is multi-valued.
14. **Q.** Find the inverse of $f(x) = (2x + 1)/3$.
    A. $y = (2x+1)/3 \rightarrow 3y = 2x + 1 \rightarrow x = (3y - 1)/2$. So $f^{-1}(x) = (3x - 1)/2$.
15. **Q.** Find the inverse of $f(x) = 1/x$.
    A. $f^{-1}(x) = 1/x$ — it is its own inverse.

Quiz Answers

1. **Answer: (C) $f^{-1}(x)$**
   Reason: The −1 superscript on f, not on x, denotes the inverse function. It is not a reciprocal.
2. **Answer: (C) one-to-one**

Reason: One-to-one — each output comes from a unique input — is the precise condition for invertibility.

3. **Answer: (B)** **$f^{-1}(x) = (x - 6)/2$**

   Reason: $y = 2x + 6 \rightarrow$ swap and solve: $y = (x - 6)/2$.

4. **Answer: (D)** **Yes — composition gives x both ways.**

   Reason: $f(g(x)) = 2\cdot(x-1)/2 + 1 = x$, and $g(f(x)) = (2x+1-1)/2 = x$. Confirmed.

5. **Answer: (C)** **$y = x$**

   Reason: Swapping x and y flips coordinates across the diagonal $y = x$.

6. **Answer: (C)** **It is not one-to-one without restriction.**

   Reason: x^2 maps both 2 and -2 to 4 — not one-to-one. Restrict to $x \geq 0$ to invert.

7. **Answer: (A)** **$C(F) = (5/9)(F - 32)$**

   Reason: Subtract 32, then multiply by 5/9 — undoes Fahrenheit conversion.

ANSWER KEY

Week 4 — Applications and Review

Practice Exercises

1. **Q.** Evaluate $f(x) = 2x + 5$ at $x = -4$.
   A. $f(-4) = -8 + 5 = -3$.
2. **Q.** Evaluate $f(x) = x^2 - 3$ at $x = 4$.
   A. $f(4) = 16 - 3 = 13$.
3. **Q.** Describe the transformation from x^2 to $(x - 1)^2 + 4$.
   A. Right 1, up 4. Vertex (1, 4).
4. **Q.** Describe the transformation from x^2 to $-(x + 2)^2 - 1$.
   A. Left 2, reflect over x-axis, down 1. Vertex (−2, −1), opens down.
5. **Q.** Find the inverse of $f(x) = 4x - 7$.
   A. $f^{-1}(x) = (x + 7)/4$.
6. **Q.** Find the inverse of $f(x) = (x - 3)/2$.
   A. $f^{-1}(x) = 2x + 3$.
7. **Q.** State domain and range of $f(x) = 1/(x + 2)$.
   A. Domain: $x \neq -2$. Range: $y \neq 0$.
8. **Q.** State domain and range of $f(x) = \sqrt{x}$.
   A. Domain: $x \geq 0$. Range: $y \geq 0$.
9. **Q.** A distance model is $d(t) = 5t + 2$. Find $d(6)$.
   A. $d(6) = 30 + 2 = 32$.
10. **Q.** For $d(t) = 5t + 2$, find the inverse and interpret.
    A. $t(d) = (d - 2)/5$; time required to reach distance d.
11. **Q.** Determine whether $f(x) = 3x^5 - x$ is even, odd, or neither.
    A. Odd (all odd powers, no constant).
12. **Q.** Determine whether $f(x) = x^2 + |x|$ is even, odd, or neither.
    A. Even.
13. **Q.** Identify the parent function of $g(x) = 2|x - 3| + 5$.
    A. $f(x) = |x|$.
14. **Q.** Identify the parent function of $g(x) = \sqrt{(x + 7)} - 2$.
    A. $f(x) = \sqrt{x}$.
15. **Q.** If $T(x) = 4x + 10$ models temperature, when does $T = 50$?
    A. $50 = 4x + 10 \rightarrow x = 10$.

Quiz Answers

1. **Answer: (D) Left 3, down 4**
   Reason: Inside $(x + 3)$ flips sign: left 3. Outside −4: down 4.
2. **Answer: (B) −3**
   Reason: $f(-1) = -1 + (-2) = -3$.

3. **Answer: (A) $f^{-1}(x) = (x + 7)/4$**

   Reason: $y = 4x - 7 \rightarrow x = 4y - 7 \rightarrow y = (x + 7)/4$.

4. **Answer: (A) All real numbers except $x = -2$**

   Reason: Denominator zero at $x = -2$; exclude.

5. **Answer: (B) $x \geq 0$**

   Reason: Square root requires non-negative input.

6. **Answer: (D) $t(d) = (d - 2)/5$**

   Reason: Solve for t: $t = (d - 2)/5$. This gives the time required to reach distance d.

7. **Answer: (D) x^4 is even; x^3 is odd.**

   Reason: Even exponents give $f(-x) = f(x)$ — x^4 is even; odd exponents give $f(-x) = -f(x)$ — x^3 is odd.

8. **Answer: (A) 3**

   Reason: $\sqrt{(13 - 4)} = \sqrt{9} = 3$.

UNIT 2

Polynomial and Rational Functions

UNIT OVERVIEW

Dive deeper into polynomials — their shape, end behaviour, roots, and factors — then unlock rational functions with their asymptotes, holes, and applications in modelling rates, optimisation, profit, and motion.

Weeks in this unit:

Week 5 — *Graphing Polynomials*
Week 6 — *Roots and Factors*
Week 7 — *Rational Functions*
Week 8 — *Applications of Polynomial and Rational Functions*

UNIT 2 · POLYNOMIAL AND RATIONAL FUNCTIONS

WEEK 5

Graphing Polynomials

Polynomials come in many shapes — straight lines, parabolas, S-curves, double-humps, and more. Learn to read a polynomial graph from its degree, leading coefficient, zeros, and multiplicity.

Learning Objectives

By the end of this week, you will be able to:

1. Identify polynomial functions and their degree.
2. Determine end behaviour using leading coefficient.
3. Find zeros and x-intercepts.
4. Analyse multiplicity of roots.
5. Sketch polynomial graphs using key features.
6. Compare factored and expanded forms.
7. Interpret turning points and graph shape.
8. Apply polynomial graphs to real-world problems.

Key Concepts

1. What is a polynomial?

A polynomial function is built from variables raised to whole-number powers, multiplied by real coefficients, and added together. The highest power is called the degree.

2. End behaviour

End behaviour describes what the graph does as x heads to positive or negative infinity. It depends on two things: whether the degree is even or odd, and whether the leading coefficient is positive or negative.

3. Zeros and multiplicity

A zero is a value of x where f of x equals zero. Multiplicity tells you how many times a factor appears, and it controls how the graph behaves at that zero.

4. Factored vs expanded form

Factored form reveals the zeros directly. Expanded form makes the degree and leading coefficient obvious. You need both.

5. Worked examples

Five worked polynomial graphing examples.

Worked Examples

Example 1. Find the zeros of $f(x) = (x - 4)(x + 3)$.

Solution: $x = 4$ and $x = -3$.

Example 2. Find the zeros of $f(x) = x^2 - 9$.

Solution: $(x - 3)(x + 3) = 0$, so $x = 3$ and $x = -3$.

Example 3. Find the zeros of $f(x) = (x - 2)^2(x + 1)$.

Solution: $x = 2$ (multiplicity 2, touches) and $x = -1$ (crosses).

UNIT 2 · POLYNOMIAL AND RATIONAL FUNCTIONS

Week 5 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Find the zeros of $f(x) = (x - 4)(x + 3)$.

2. Find the zeros of $f(x) = x^2 - 9$.

3. Find the zeros of $f(x) = (x - 2)^2(x + 1)$.

4. State the end behaviour of $f(x) = x^4$.

5. State the end behaviour of $f(x) = -x^3$.

6. What is the degree of $f(x) = 2x^5 - x^3 + 7$?

7. What is the leading coefficient of $f(x) = -4x^3 + 2x - 1$?

8. Sketch features of $f(x) = (x - 1)(x - 2)(x + 3)$.

9. How many turning points can a degree-5 polynomial have at most?

10. Find the vertex of $f(x) = x^2 - 6x + 5$.

11. Find the maximum of $P(x) = -x^2 + 10x$.

12. For $R(x) = -x^2 + 8x$, find zeros.

13. Why is $f(x) = (x - 1)^2(x + 2)^3$ called a "kissing" graph at $x = 1$?

14. At $x = -2$ in the previous problem, what happens?

15. A revenue model is $R(x) = -x^2 + 12x$. Find break-even points and maximum revenue.

UNIT 2 · POLYNOMIAL AND RATIONAL FUNCTIONS

Week 5 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. What is the degree of $f(x) = 3x^4 - 2x^2 + 5$?

(A) 4
(B) 3
(C) 5
(D) 2
Your answer: ____________

2. A polynomial of degree 5 with a positive leading coefficient has what end behaviour?

(A) Both ends down
(B) Left down, right up
(C) Left up, right down
(D) Both ends up
Your answer: ____________

3. Find the zeros of $f(x) = (x - 4)(x + 2)(x - 1)$.

(A) $x = 4, -2, -1$
(B) $x = 4, -2, 1$
(C) $x = 4, 2, 1$
(D) $x = -4, 2, -1$
Your answer: ____________

4. At a zero with multiplicity 2, the graph…

(A) Touches the x-axis and turns
(B) Disappears
(C) Crosses the x-axis
(D) Becomes infinite
Your answer: ____________

5. Which form makes the zeros most obvious?

(A) Expanded form
(B) Standard form
(C) Factored form
(D) Vertex form
Your answer: ____________

6. For $f(x) = x^4$, both ends of the graph point…

(A) right
(B) left

(C) up
(D) down
Your answer: ____________

7. A polynomial of degree 4 has at most how many turning points?

(A) 4
(B) 3
(C) 5
(D) 2
Your answer: ____________

8. For $P(x) = -x^2 + 8x$, what is the maximum value?

(A) 16 at x = 8
(B) 0 at x = 0
(C) 8 at x = 4
(D) 16 at x = 4
Your answer: ____________

DID YOU KNOW?

Engineers use degree-3 (cubic) and degree-5 (quintic) polynomial splines to design the smooth curves you see in every car, every aeroplane wing, and every Pixar animation. The trick is that a cubic has exactly enough freedom to match a position, a slope, and a curvature at each endpoint — so you can stitch them together without anyone noticing the seams.

UNIT 2 · POLYNOMIAL AND RATIONAL FUNCTIONS

WEEK 6

Roots and Factors

Roots and factors are two sides of the same coin. Learn the factor theorem, the remainder theorem, and the three workhorse factoring techniques: GCF, trinomials, and grouping.

Learning Objectives

By the end of this week, you will be able to:

1. Define roots (zeros) of a polynomial.
2. Connect roots to factors of a polynomial.
3. Apply the factor theorem.
4. Factor polynomial expressions using basic techniques.
5. Solve polynomial equations by factoring.
6. Identify multiplicity of roots.
7. Interpret roots graphically.
8. Apply factoring to real-world problems.

Key Concepts

1. Roots and factors are partners

A root is a value of x where f of x equals zero. A factor is an expression that divides the polynomial exactly. If x equals a is a root, then x minus a is a factor.

2. The Factor and Remainder Theorems

The factor theorem says: x minus a is a factor of f if and only if f of a equals zero. The remainder theorem says: when you divide f of x by x minus a, the remainder is f of a.

3. Three factoring techniques

There are three workhorse techniques you must master: greatest common factor, trinomial factoring, and factoring by grouping.

4. Solving by factoring

To solve a polynomial equation, get everything on one side equal to zero, factor, and use the zero product property.

5. Worked examples

Five factoring examples.

Worked Examples

Example 1. Find the roots of $f(x) = (x - 7)(x + 2)$.

Solution: $x = 7$ and $x = -2$.

Example 2. Find the roots of $f(x) = x^2 - 16$.

Solution: $(x - 4)(x + 4) = 0 \rightarrow x = 4, x = -4$.

Example 3. Factor $x^2 + 9x + 20$.

Solution: $(x + 4)(x + 5)$ (since $4 \cdot 5 = 20$ and $4 + 5 = 9$).

UNIT 2 · POLYNOMIAL AND RATIONAL FUNCTIONS

Week 6 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Find the roots of $f(x) = (x - 7)(x + 2)$.

2. Find the roots of $f(x) = x^2 - 16$.

3. Factor $x^2 + 9x + 20$.

4. Factor $x^2 - x - 12$.

5. Factor $x^2 - 49$.

6. Factor $6x^3 + 9x^2$.

7. Solve $x^2 - 5x = 0$.

8. Solve $x^2 + 2x - 8 = 0$.

9. Verify $(x - 1)$ is a factor of $f(x) = x^3 - 1$.

10. By remainder theorem, find the remainder of $f(x) = x^2 - 3x + 1$ divided by $(x - 4)$.

11. Factor $x^3 - 8$ (difference of cubes).

12. Factor by grouping: $x^3 + 3x^2 + 2x + 6$.

13. A profit $P(x) = x^2 - 5x - 6$. Find break-even points.

14. A height $h(x) = -x^2 + 6x$ reaches the ground when $h = 0$. Find when.

15. How does the factor $(x - 3)^2$ affect the graph at $x = 3$?

UNIT 2 · POLYNOMIAL AND RATIONAL FUNCTIONS

Week 6 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. If $x = 5$ is a root of f, then which is a factor?

(A) $(x - 1)$
(B) $(x + 5)$
(C) $(x - 5)$
(D) $(5 - x)/x$
Your answer: ____________

2. Factor $x^2 - 25$.

(A) $(x - 5)^2$
(B) $(x - 5)(x + 5)$
(C) $x(x - 25)$
(D) $(x - 25)(x + 1)$
Your answer: ____________

3. Solve $x^2 - 7x + 12 = 0$.

(A) $x = 7, x = 12$
(B) $x = 1, x = 12$
(C) $x = 3, x = 4$
(D) $x = -3, x = -4$
Your answer: ____________

4. By the factor theorem, $(x - 2)$ is a factor of $f(x) = x^2 + x - 6$ if and only if…

(A) $f(0) = 2$
(B) $x^2 + x - 6$ has even degree
(C) $f(2) = 0$
(D) $f(-2) = 0$
Your answer: ____________

5. By the remainder theorem, when $f(x) = x^2 + 3x - 1$ is divided by $(x - 2)$, the remainder is…

(A) 5
(B) 9
(C) 0
(D) 3
Your answer: ____________

6. Factor $4x^2 + 8x$ using the GCF.

(A) $x(4x + 8)$
(B) $4x(x + 2)$

(C) $4(x^2 + 2x)$
(D) $2x(2x + 4)$
Your answer: ____________

7. Solve $x^2 - 5x = 0$.

(A) $x = -5$ only
(B) $x = 0, x = 5$
(C) $x = 5, x = -5$
(D) $x = 0$ only
Your answer: ____________

8. A projectile height is $h(x) = -x^2 + 4x$. When does it return to the ground ($h = 0$)?

(A) $x = 2$
(B) $x = 4$ only
(C) $x = 0$ and $x = 4$
(D) $x = -4$ and $x = 0$
Your answer: ____________

DID YOU KNOW?

Cryptography in the 21st century is built on a fact about factoring: while it is easy to multiply two huge prime numbers together, it is extraordinarily hard to factor the result back into those primes. Every secure website connection (HTTPS) relies on this. The maths you are doing this week — finding factors — is, at industrial scale, what protects the global internet.

UNIT 2 · POLYNOMIAL AND RATIONAL FUNCTIONS

WEEK 7

Rational Functions

Rational functions are ratios of polynomials. They bring in asymptotes, holes, and forbidden inputs — the most colourful graphs in precalculus.

Learning Objectives

By the end of this week, you will be able to:

1. Define a rational function.
2. Identify restrictions on the domain.
3. Simplify rational expressions correctly.
4. Determine vertical asymptotes and holes.
5. Identify horizontal asymptotes (basic level).
6. Solve simple rational equations.
7. Interpret graphs of rational functions.
8. Apply rational functions to real-world problems.

Key Concepts

1. What is a rational function?

A rational function is a ratio of two polynomials. The denominator can never equal zero, which creates restrictions on the domain.

2. Vertical asymptotes and holes

A vertical asymptote occurs where the denominator equals zero and does not cancel. A hole occurs where a factor cancels from numerator and denominator.

3. Horizontal asymptotes

Horizontal asymptotes describe the long-run behaviour of a rational function. They depend on the degrees of the top and bottom.

4. Simplifying and solving

To simplify, factor and cancel common factors. To solve, find a common denominator and check for extraneous solutions.

5. Worked examples

Five rational function examples.

Worked Examples

Example 1. State the domain of $f(x) = 1/(x - 5)$.

Solution: All real numbers except $x = 5$.

Example 2. State the domain of $f(x) = 3/(x^2 - 4)$.

Solution: All real numbers except $x = 2$ and $x = -2$.

Example 3. Simplify $(x^2 - 16)/(x - 4)$.

Solution: $(x - 4)(x + 4)/(x - 4) = x + 4, x \neq 4$.

UNIT 2 · POLYNOMIAL AND RATIONAL FUNCTIONS

Week 7 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. State the domain of $f(x) = 1/(x - 5)$.

2. State the domain of $f(x) = 3/(x^2 - 4)$.

3. Simplify $(x^2 - 16)/(x - 4)$.

4. Simplify $(x^2 + 5x + 6)/(x + 2)$.

5. Find vertical and horizontal asymptotes of $f(x) = 1/x$.

6. Find vertical and horizontal asymptotes of $f(x) = (x + 1)/(x - 2)$.

7. Find the horizontal asymptote of $f(x) = (4x^2 + 1)/(2x^2 + 5)$.

8. Find the horizontal asymptote of $f(x) = 7/(x^3 + 1)$.

9. Does $f(x) = x^2/(x + 1)$ have a horizontal asymptote?

10. Where is the hole in $f(x) = (x + 3)/(x^2 + 3x)$?

11. A rate model $R(x) = 100/(x + 2)$. Domain (for $x > 0$)?

12. Evaluate $R(3)$ for $R(x) = 60/x$.

13. As x grows large in $R(x) = 60/x$, what happens to $R(x)$?

14. Solve $3/x = 6$.

15. Solve $1/(x - 1) = 2$.

UNIT 2 · POLYNOMIAL AND RATIONAL FUNCTIONS

Week 7 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. What is the domain of $f(x) = 1/(x - 5)$?

(A) All real numbers except $x = -5$
(B) All real numbers except $x = 5$
(C) $x > 5$
(D) $x < 5$
Your answer: ____________

2. Find the vertical asymptote of $f(x) = 3/(x + 2)$.

(A) $x = 3$
(B) $y = 0$
(C) $x = -2$
(D) $x = 2$
Your answer: ____________

3. Find the horizontal asymptote of $f(x) = 5/(x^2 + 1)$.

(A) $y = 5$
(B) $y = 1$
(C) No horizontal asymptote
(D) $y = 0$
Your answer: ____________

4. Find the horizontal asymptote of $f(x) = (3x + 1)/(x + 7)$.

(A) $y = 7$
(B) $y = 0$
(C) $y = 3$
(D) $y = 1$
Your answer: ____________

5. What happens to the graph of $f(x) = (x - 2)(x + 3)/(x - 2)$ at $x = 2$?

(A) Hole
(B) x-intercept
(C) Vertical asymptote
(D) Horizontal asymptote
Your answer: ____________

6. Simplify $(x^2 - 16)/(x - 4)$ (note the restriction).

(A) $x + 16$
(B) $x^2, x \neq 4$
(C) $x - 4$

(D) $x + 4, x \neq 4$

Your answer: ____________

7. What is the domain of $f(x) = 3/(x^2 - 4)$?

(A) x 2

(B) All real numbers except $x = 0$

(C) $x > 2$

(D) All real numbers except $x = \pm 2$

Your answer: ____________

8. A rate model is $R(x) = 100/(x + 2)$. What does the vertical asymptote at $x = -2$ mean physically (if x is positive)?

(A) It is the minimum rate.

(B) The rate becomes 100.

(C) The model is mathematically undefined there, often outside the meaningful range.

(D) It is the maximum rate.

Your answer: ____________

DID YOU KNOW?

When pharmacists calculate drug dosages, they use rational functions all the time. The Michaelis-Menten equation that governs how fast enzymes break down medication has the form $v = V \cdot x/(K + x)$ — a rational function whose horizontal asymptote tells you the maximum possible reaction rate. So rational functions literally power the dose calculations behind every prescription.

UNIT 2 · POLYNOMIAL AND RATIONAL FUNCTIONS

WEEK 8

Applications of Polynomial and Rational Functions

Time to put it to work. Use polynomials to model profit, area, and motion; use rational functions to model rates, density, and efficiency. Master the art of choosing the right function for the situation.

Learning Objectives

By the end of this week, you will be able to:

1. Apply polynomial functions to real-world scenarios.
2. Apply rational functions to rate-based problems.
3. Interpret graphs in applied contexts.
4. Solve optimisation problems using polynomials.
5. Model real-world situations using functions.
6. Identify key features (zeros, asymptotes) in context.
7. Compare polynomial vs rational models.
8. Analyse and interpret data using function models.

Key Concepts

1. Choosing the right function

Polynomials and rational functions model different kinds of situations. Polynomials work for growth, area, and profit. Rational functions work for rates, ratios, and inverse relationships.

2. Optimisation with polynomials

Many business and engineering problems ask for maximum or minimum values. For a quadratic, find the vertex.

3. Rate problems with rational functions

Speed equals distance divided by time. Work and density and concentration all follow similar rational forms.

4. Interpreting graphs in context

When you see a graph in a word problem, ask: what do the zeros mean? what do the turning points mean? what do the asymptotes mean?

5. Worked examples

Five applied examples.

Worked Examples

Example 1. Find the maximum of $P(x) = -x^2 + 6x - 5$.

Solution: Vertex at $x = 3$, $P(3) = 4$. Maximum is 4.

Example 2. Find break-even points: $P(x) = x^2 - 7x + 12$.

Solution: $(x - 3)(x - 4) = 0 \rightarrow x = 3$ and $x = 4$.

Example 3. $R(x) = -2x^2 + 10x$. Find maximum revenue.

Solution: Vertex at $x = 5/2 = 2.5$, $R(2.5) = 12.5$. Maximum revenue 12.5.

UNIT 2 · POLYNOMIAL AND RATIONAL FUNCTIONS

Week 8 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Find the maximum of $P(x) = -x^2 + 6x - 5$.

2. Find break-even points: $P(x) = x^2 - 7x + 12$.

3. $R(x) = -2x^2 + 10x$. Find maximum revenue.

4. Find vertex of $f(x) = -x^2 + 4x + 1$.

5. Speed $S(x) = 60/x$. Find $S(2)$, $S(4)$, $S(10)$.

6. For $R(x) = 100/x$, what happens as x grows very large?

7. $f(x) = 100/(x + 5)$. State domain.

8. $P(x) = -x^2 + 10x - 16$. Find break-even and maximum profit.

9. Choose model type: cost per item as quantity increases (with fixed cost).

10. Choose model type: profit as units sold change.

11. For $f(x) = (x - 3)(x + 4)$ what are the zeros?

12. For $f(x) = (x + 2)/(x - 3)$, find vertical and horizontal asymptotes.

13. Compare $f(x) = x^2$ and $g(x) = 1/x$: how do their graphs behave near $x = 0$?

14. A box has length x, width $x + 2$, height 5. Express volume as a polynomial.

15. For $V(x) = 5x^2 + 10x$, find $V(3)$.

UNIT 2 · POLYNOMIAL AND RATIONAL FUNCTIONS

Week 8 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. For $P(x) = -x^2 + 8x$, what is the maximum value?

- **(A)** 8 at $x = 4$
- **(B)** 0 at $x = 0$
- **(C)** 64 at $x = 8$
- **(D)** 16 at $x = 4$

Your answer: ____________

2. For $P(x) = x^2 - 8x + 12$, find the break-even points.

- **(A)** $x = 2$ and $x = 6$
- **(B)** $x = 0$ and $x = 8$
- **(C)** $x = -2$ and $x = -6$
- **(D)** $x = 4$ and $x = 6$

Your answer: ____________

3. A speed model $S(x) = 120/x$. What is $S(6)$?

- **(A)** 60
- **(B)** 40
- **(C)** 12
- **(D)** 20

Your answer: ____________

4. What does the vertical asymptote of $R(x) = 100/(x - 2)$ mean physically?

- **(A)** The maximum rate.
- **(B)** The break-even point.
- **(C)** The minimum rate.
- **(D)** The model is undefined/unstable at $x = 2$.

Your answer: ____________

5. For $f(x) = (x^2 - 1)/(x - 1)$, what happens at $x = 1$?

- **(A)** Nothing
- **(B)** A vertical asymptote
- **(C)** A hole
- **(D)** A maximum

Your answer: ____________

6. $P(x) = -2x^2 + 12x - 10$. Find the maximum profit.

- **(A)** 8 at $x = 3$
- **(B)** 6 at $x = 3$
- **(C)** 10 at $x = 0$

(D) 12 at $x = 6$

Your answer: ____________

7. For $R(x) = (5x + 1)/(x + 2)$, what is the horizontal asymptote (long-run behaviour)?

(A) $y = 2$
(B) $y = 5$
(C) $y = 1$
(D) No horizontal asymptote

Your answer: ____________

8. Which type best models how concentration changes as solvent is added?

(A) Polynomial
(B) Rational
(C) Exponential
(D) Logarithmic

Your answer: ____________

DID YOU KNOW?

NASA models rocket trajectories with polynomials but uses rational functions to predict atmospheric drag, where air density changes with altitude. The combination — polynomials for the smooth motion, rational functions for the resistance — is exactly the kind of multi-model thinking that this unit is preparing you to do.

ANSWER KEY

Unit 2 · Polynomial and Rational Functions

This answer key covers every practice exercise and quiz question from Unit 2. For full step-by-step solutions to randomised practice generators (separate from the worksheet exercises printed here), refer to the BemandaSTEM Precalculus app.

ANSWER KEY

Week 5 — Graphing Polynomials

Practice Exercises

1. **Q.** Find the zeros of $f(x) = (x - 4)(x + 3)$.
   A. $x = 4$ and $x = -3$.
2. **Q.** Find the zeros of $f(x) = x^2 - 9$.
   A. $(x - 3)(x + 3) = 0$, so $x = 3$ and $x = -3$.
3. **Q.** Find the zeros of $f(x) = (x - 2)^2(x + 1)$.
   A. $x = 2$ (multiplicity 2, touches) and $x = -1$ (crosses).
4. **Q.** State the end behaviour of $f(x) = x^4$.
   A. Both ends rise (up, up).
5. **Q.** State the end behaviour of $f(x) = -x^3$.
   A. Left up, right down.
6. **Q.** What is the degree of $f(x) = 2x^5 - x^3 + 7$?
   A. 5.
7. **Q.** What is the leading coefficient of $f(x) = -4x^3 + 2x - 1$?
   A. -4.
8. **Q.** Sketch features of $f(x) = (x - 1)(x - 2)(x + 3)$.
   A. Zeros at $x = 1, 2, -3$ (all cross). Degree 3, positive leading coefficient: left down, right up.
9. **Q.** How many turning points can a degree-5 polynomial have at most?
   A. 4.
10. **Q.** Find the vertex of $f(x) = x^2 - 6x + 5$.
    A. $x = 3$, $y = 9 - 18 + 5 = -4$. Vertex $(3, -4)$.
11. **Q.** Find the maximum of $P(x) = -x^2 + 10x$.
    A. Vertex $x = 5$, $P(5) = 25$. Maximum profit 25.
12. **Q.** For $R(x) = -x^2 + 8x$, find zeros.
    A. $x(-x + 8) = 0 \rightarrow x = 0$ and $x = 8$.
13. **Q.** Why is $f(x) = (x - 1)^2(x + 2)^3$ called a "kissing" graph at $x = 1$?
    A. Multiplicity 2 at $x = 1$ means the graph touches and turns rather than crosses — it kisses the axis.
14. **Q.** At $x = -2$ in the previous problem, what happens?
    A. Multiplicity 3 is odd; the graph crosses the x-axis (with a flattened S-shape).
15. **Q.** A revenue model is $R(x) = -x^2 + 12x$. Find break-even points and maximum revenue.
    A. Zeros at $x = 0$ and $x = 12$. Vertex at $x = 6$, $R(6) = 36$.

Quiz Answers

1. **Answer: (A) 4**
   Reason: Degree is the highest exponent — here, 4.
2. **Answer: (B) Left down, right up**

Reason: Odd degree, positive leading coefficient: classic cubic-like S — left down, right up.

3. **Answer: (B) $x = 4, -2, 1$**

   Reason: Set each factor to zero: $x = 4$, $x = -2$, $x = 1$.

4. **Answer: (A) Touches the x-axis and turns**

   Reason: Even multiplicity → the graph touches and bounces off, like a parabola landing.

5. **Answer: (C) Factored form**

   Reason: Factored form sets each factor equal to zero to read off the roots directly.

6. **Answer: (C) up**

   Reason: Even degree with positive leading coefficient — both ends rise.

7. **Answer: (B) 3**

   Reason: Degree n has at most $n - 1$ turning points; $4 - 1 = 3$.

8. **Answer: (D) 16 at $x = 4$**

   Reason: Vertex at $x = -b/2a = -8/(-2) = 4$. $P(4) = -16 + 32 = 16$.

ANSWER KEY

Week 6 — Roots and Factors

Practice Exercises

1. **Q.** Find the roots of $f(x) = (x - 7)(x + 2)$.

   A. $x = 7$ and $x = -2$.

2. **Q.** Find the roots of $f(x) = x^2 - 16$.

   A. $(x - 4)(x + 4) = 0 \rightarrow x = 4, x = -4$.

3. **Q.** Factor $x^2 + 9x + 20$.

   A. $(x + 4)(x + 5)$ (since $4 \cdot 5 = 20$ and $4 + 5 = 9$).

4. **Q.** Factor $x^2 - x - 12$.

   A. $(x - 4)(x + 3)$.

5. **Q.** Factor $x^2 - 49$.

   A. $(x - 7)(x + 7)$.

6. **Q.** Factor $6x^3 + 9x^2$.

   A. $3x^2(2x + 3)$.

7. **Q.** Solve $x^2 - 5x = 0$.

   A. $x(x - 5) = 0 \rightarrow x = 0, x = 5$.

8. **Q.** Solve $x^2 + 2x - 8 = 0$.

   A. $(x + 4)(x - 2) = 0 \rightarrow x = -4, x = 2$.

9. **Q.** Verify $(x - 1)$ is a factor of $f(x) = x^3 - 1$.

   A. $f(1) = 1 - 1 = 0$ ✓. So $(x - 1)$ is a factor.

10. **Q.** By remainder theorem, find the remainder of $f(x) = x^2 - 3x + 1$ divided by $(x - 4)$.

    A. $f(4) = 16 - 12 + 1 = 5$.

11. **Q.** Factor $x^3 - 8$ (difference of cubes).

    A. $(x - 2)(x^2 + 2x + 4)$.

12. **Q.** Factor by grouping: $x^3 + 3x^2 + 2x + 6$.

    A. $x^2(x + 3) + 2(x + 3) = (x + 3)(x^2 + 2)$.

13. **Q.** A profit $P(x) = x^2 - 5x - 6$. Find break-even points.

    A. $(x - 6)(x + 1) = 0 \rightarrow x = 6$ or $x = -1$. Break-even at $x = 6$ (positive units).

14. **Q.** A height $h(x) = -x^2 + 6x$ reaches the ground when $h = 0$. Find when.

    A. $-x(x - 6) = 0 \rightarrow x = 0$ (launch) or $x = 6$ (landing).

15. **Q.** How does the factor $(x - 3)^2$ affect the graph at $x = 3$?

    A. Multiplicity 2 — the graph touches x-axis and bounces, does not cross.

Quiz Answers

1. **Answer: (C) $(x - 5)$**

   Reason: A root at $x = a$ corresponds to the factor $(x - a)$. Here, $(x - 5)$.

2. **Answer: (B) $(x - 5)(x + 5)$**

   Reason: Difference of squares: $x^2 - 5^2 = (x - 5)(x + 5)$.

3. **Answer: (C) $x = 3, x = 4$**

   Reason: Factor: $(x - 3)(x - 4) = 0$. So $x = 3$ or $x = 4$.

4. **Answer: (C) $f(2) = 0$**

   Reason: Factor theorem: $(x - a)$ divides f if and only if $f(a) = 0$. Here $f(2) = 4 + 2 - 6 = 0$ — yes.

5. **Answer: (B) 9**

   Reason: $f(2) = 4 + 6 - 1 = 9$.

6. **Answer: (B) $4x(x + 2)$**

   Reason: GCF is $4x$. Factor: $4x(x + 2)$.

7. **Answer: (B) $x = 0, x = 5$**

   Reason: Factor: $x(x - 5) = 0$. So $x = 0$ or $x = 5$.

8. **Answer: (C) $x = 0$ and $x = 4$**

   Reason: Set $-x^2 + 4x = 0 \rightarrow -x(x - 4) = 0 \rightarrow x = 0$ (launch) or $x = 4$ (return).

ANSWER KEY

Week 7 — Rational Functions

Practice Exercises

1. **Q.** State the domain of $f(x) = 1/(x - 5)$.

   A. All real numbers except $x = 5$.

2. **Q.** State the domain of $f(x) = 3/(x^2 - 4)$.

   A. All real numbers except $x = 2$ and $x = -2$.

3. **Q.** Simplify $(x^2 - 16)/(x - 4)$.

   A. $(x - 4)(x + 4)/(x - 4) = x + 4$, $x \neq 4$.

4. **Q.** Simplify $(x^2 + 5x + 6)/(x + 2)$.

   A. $(x + 2)(x + 3)/(x + 2) = x + 3$, $x \neq -2$.

5. **Q.** Find vertical and horizontal asymptotes of $f(x) = 1/x$.

   A. Vertical: $x = 0$; Horizontal: $y = 0$.

6. **Q.** Find vertical and horizontal asymptotes of $f(x) = (x + 1)/(x - 2)$.

   A. Vertical: $x = 2$; Horizontal: $y = 1$.

7. **Q.** Find the horizontal asymptote of $f(x) = (4x^2 + 1)/(2x^2 + 5)$.

   A. Same degree; $y = 4/2 = 2$.

8. **Q.** Find the horizontal asymptote of $f(x) = 7/(x^3 + 1)$.

   A. Top degree (0) < bottom (3): $y = 0$.

9. **Q.** Does $f(x) = x^2/(x + 1)$ have a horizontal asymptote?

   A. No — top degree > bottom degree.

10. **Q.** Where is the hole in $f(x) = (x + 3)/(x^2 + 3x)$?

    A. Factor: $(x + 3)/(x(x + 3)) = 1/x$ with $x \neq -3$. Hole at $x = -3$; vertical asymptote at $x = 0$.

11. **Q.** A rate model $R(x) = 100/(x + 2)$. Domain (for $x > 0$)?

    A. All positive x; mathematical restriction $x \neq -2$ lies outside the physical domain.

12. **Q.** Evaluate $R(3)$ for $R(x) = 60/x$.

    A. $R(3) = 20$.

13. **Q.** As x grows large in $R(x) = 60/x$, what happens to $R(x)$?

    A. $R(x)$ approaches 0 — horizontal asymptote at $y = 0$.

14. **Q.** Solve $3/x = 6$.

    A. Multiply both sides by x: $3 = 6x \rightarrow x = 1/2$.

15. **Q.** Solve $1/(x - 1) = 2$.

    A. $1 = 2(x - 1) \rightarrow 1 = 2x - 2 \rightarrow x = 3/2$.

Quiz Answers

1. **Answer: (B) All real numbers except $x = 5$**

   Reason: Denominator is zero at $x = 5$; the function is undefined there.

2. **Answer: (C) $x = -2$**

Reason: Set denominator equal to zero: $x + 2 = 0 \rightarrow x = -2$. Numerator non-zero, so it is a vertical asymptote.

3. **Answer: (D) $y = 0$**

   Reason: Numerator degree (0) is less than denominator degree (2). Horizontal asymptote at $y = 0$.

4. **Answer: (C) $y = 3$**

   Reason: Degrees equal (both 1); ratio of leading coefficients is $3/1 = 3$.

5. **Answer: (A) Hole**

   Reason: The factor $(x - 2)$ cancels — discontinuity is removable, producing a hole at $x = 2$.

6. **Answer: (D) $x + 4, x \neq 4$**

   Reason: Factor: $(x - 4)(x + 4)/(x - 4) = x + 4$, with $x \neq 4$.

7. **Answer: (D) All real numbers except $x = \pm 2$**

   Reason: $x^2 - 4 = 0$ when $x = \pm 2$; exclude both.

8. **Answer: (C) The model is mathematically undefined there, often outside the meaningful range.**

   Reason: For positive x, $x = -2$ is outside the physical range. Mathematically, it would mean a discontinuity — practically, an irrelevant edge of the model.

ANSWER KEY

Week 8 — Applications of Polynomial and Rational Functions

Practice Exercises

1. **Q.** Find the maximum of $P(x) = -x^2 + 6x - 5$.
   A. Vertex at $x = 3$, $P(3) = 4$. Maximum is 4.
2. **Q.** Find break-even points: $P(x) = x^2 - 7x + 12$.
   A. $(x - 3)(x - 4) = 0 \rightarrow x = 3$ and $x = 4$.
3. **Q.** $R(x) = -2x^2 + 10x$. Find maximum revenue.
   A. Vertex at $x = 5/2 = 2.5$, $R(2.5) = 12.5$. Maximum revenue 12.5.
4. **Q.** Find vertex of $f(x) = -x^2 + 4x + 1$.
   A. $x = -4/(-2) = 2$, $f(2) = -4 + 8 + 1 = 5$. Vertex $(2, 5)$.
5. **Q.** Speed $S(x) = 60/x$. Find $S(2)$, $S(4)$, $S(10)$.
   A. $S(2) = 30$; $S(4) = 15$; $S(10) = 6$.
6. **Q.** For $R(x) = 100/x$, what happens as x grows very large?
   A. $R(x) \rightarrow 0$; horizontal asymptote $y = 0$.
7. **Q.** $f(x) = 100/(x + 5)$. State domain.
   A. All real x except $x = -5$.
8. **Q.** $P(x) = -x^2 + 10x - 16$. Find break-even and maximum profit.
   A. Factor: $-(x^2 - 10x + 16) = -(x - 2)(x - 8)$. Break-even at $x = 2$ and $x = 8$. Vertex $x = 5$, $P(5) = -25 + 50 - 16 = 9$. Maximum profit 9.
9. **Q.** Choose model type: cost per item as quantity increases (with fixed cost).
   A. Rational — cost = (fixed + variable)/quantity.
10. **Q.** Choose model type: profit as units sold change.
    A. Polynomial (typically quadratic).
11. **Q.** For $f(x) = (x - 3)(x + 4)$ what are the zeros?
    A. $x = 3$ and $x = -4$.
12. **Q.** For $f(x) = (x + 2)/(x - 3)$, find vertical and horizontal asymptotes.
    A. Vertical: $x = 3$; Horizontal: $y = 1$.
13. **Q.** Compare $f(x) = x^2$ and $g(x) = 1/x$: how do their graphs behave near $x = 0$?
    A. f is continuous; g has a vertical asymptote.
14. **Q.** A box has length x, width $x + 2$, height 5. Express volume as a polynomial.
    A. $V(x) = 5x(x + 2) = 5x^2 + 10x$.
15. **Q.** For $V(x) = 5x^2 + 10x$, find $V(3)$.
    A. $V(3) = 45 + 30 = 75$.

Quiz Answers

1. **Answer: (D) 16 at $x = 4$**
   Reason: Vertex at $x = -8/(-2) = 4$. $P(4) = -16 + 32 = 16$.
2. **Answer: (A) $x = 2$ and $x = 6$**

Reason: Factor: $(x - 2)(x - 6) = 0 \to x = 2$ and $x = 6$.

3. **Answer: (D) 20**

   Reason: $S(6) = 120/6 = 20$.

4. **Answer: (D) The model is undefined/unstable at $x = 2$.**

   Reason: A vertical asymptote represents a value where the system breaks down — the model gives no meaningful answer there.

5. **Answer: (C) A hole**

   Reason: $(x - 1)$ cancels from top and bottom — removable discontinuity, so a hole at $x = 1$.

6. **Answer: (A) 8 at $x = 3$**

   Reason: Vertex $x = -12/(-4) = 3$. $P(3) = -18 + 36 - 10 = 8$.

7. **Answer: (B) $y = 5$**

   Reason: Same degree top and bottom; ratio of leading coefficients = $5/1 = 5$.

8. **Answer: (B) Rational**

   Reason: Concentration = solute / solution — a ratio, so rational.

UNIT 3

Exponential and Logarithmic Functions

UNIT OVERVIEW

Meet the functions that govern population growth, radioactive decay, compound interest, sound intensity, and earthquake magnitude. Master exponentials, logarithms, their inverse partnership, and the real-world models built on them.

Weeks in this unit:

Week 9 — *Exponential Growth and Decay*
Week 10 — *Logarithmic Properties and Equations*
Week 11 — *Solving Exponential and Logarithmic Equations*
Week 12 — *Applications to Real-Life Problems*

UNIT 3 · EXPONENTIAL AND LOGARITHMIC FUNCTIONS

WEEK 9

Exponential Growth and Decay

Exponential functions describe phenomena where the rate of change is proportional to the current value — populations expanding, radioactive isotopes decaying, savings compounding. Small exponents create huge effects.

Learning Objectives

By the end of this week, you will be able to:

1. Define exponential growth and decay.
2. Write and interpret exponential models.
3. Identify growth and decay constants.
4. Solve basic exponential equations.
5. Analyse graphs of exponential functions.
6. Determine doubling time and half-life.
7. Apply exponential models to real-world problems.
8. Compare exponential growth vs decay behaviour.

Key Concepts

1. What is an exponential function?

An exponential function has the form f of x equals a times b to the x. The variable x is in the exponent — that is what makes the function exponential.

2. Continuous growth and the number e

When growth happens continuously rather than in discrete steps, we use the natural exponential function — f of t equals a times e to the k t. The constant e is approximately 2.718.

3. Doubling time and half-life

Doubling time is how long a growing quantity takes to double. Half-life is how long a decaying quantity takes to fall by half. Both depend only on the rate, not the initial amount.

4. Graphs of exponential functions

Exponential growth curves upward more and more steeply. Exponential decay falls toward zero, never reaching it. Both pass through the y-intercept at the initial value.

5. Worked examples

Five worked exponential examples.

Worked Examples

Example 1. Evaluate $f(x) = 3 \cdot 2^x$ at $x = 3$.

Solution: $f(3) = 3 \cdot 8 = 24$.

Example 2. Evaluate $f(x) = 100 \cdot 0.8^x$ at $x = 2$.

Solution: $f(2) = 100 \cdot 0.64 = 64$.

Example 3. Is $f(x) = 5 \cdot 1.2^x$ growth or decay?

Solution: Base $1.2 > 1$ — growth.

UNIT 3 · EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Week 9 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Evaluate $f(x) = 3 \cdot 2^x$ at $x = 3$.

2. Evaluate $f(x) = 100 \cdot 0.8^x$ at $x = 2$.

3. Is $f(x) = 5 \cdot 1.2^x$ growth or decay?

4. Is $f(x) = 10 \cdot 0.6^x$ growth or decay?

5. For $P(t) = 200 \cdot e^{0.05t}$, what is the initial population and the continuous rate?

6. Estimate doubling time at 2% growth using Rule of 70.

7. Estimate doubling time at 10% growth.

8. A bacteria culture $P(t) = 500 \cdot 1.1^t$. Find P(3).

9. A drug has half-life 4 hours. Starting with 200 mg, how much remains after 12 hours?

10. A radioisotope has half-life 5 years. After 20 years, what fraction remains?

11. For $f(x) = 2^x$, find f(0), f(1), f(2), f(3).

12. For $f(x) = (1/2)^x$, find f(0), f(1), f(2), f(3).

13. £1000 invested at 6% compounded annually. Write the model and find value after 5 years.

14. Which grows faster: $f(x) = 2^x$ or $g(x) = 1.5^x$?

15. For $P(t) = 800 \cdot e^{-0.1t}$, is it growth or decay? Find P(10).

UNIT 3 · EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Week 9 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. Which function represents exponential growth?

(A) $f(x) = 5 \cdot 2^x$
(B) $f(x) = 5x^2$
(C) $f(x) = 5/x$
(D) $f(x) = 5 \cdot 0.5^x$
Your answer: ____________

2. For $f(x) = 100 \cdot 0.8^x$, find f(0).

(A) 100
(B) 0
(C) 1
(D) 80
Your answer: ____________

3. A population $P(t) = 500 \cdot e^{0.02t}$. What is the continuous growth rate?

(A) 500%
(B) 2%
(C) 0.5%
(D) 20%
Your answer: ____________

4. Using the Rule of 70, what is the doubling time at 5% growth?

(A) 70 years
(B) 5 years
(C) 35 years
(D) 14 years
Your answer: ____________

5. A drug has a half-life of 6 hours. After 18 hours, what fraction remains?

(A) 1/8
(B) 1/3
(C) 1/18
(D) 1/6
Your answer: ____________

6. The number e is approximately…

(A) 2.718
(B) 3.142
(C) 1.618

(D) 1.414

Your answer: ____________

7. For an exponential decay function $f(x) = a \cdot b^x$, the base b satisfies…

(A) $b > 1$

(B) $b < 0$

(C) $0 < b < 1$

(D) $b = 1$

Your answer: ____________

8. For $f(x) = 3 \cdot 2^x$, what is f(4)?

(A) 12

(B) 9

(C) 24

(D) 48

Your answer: ____________

DID YOU KNOW?

A single grain of rice doubled 64 times — once for each square of a chessboard — would weigh more than the entire annual rice harvest of the Earth a thousand times over. That is the punchline of an ancient legend, and it is also why compound interest is sometimes called the eighth wonder of the world: exponential growth is genuinely hard for human intuition to grasp.

UNIT 3 · EXPONENTIAL AND LOGARITHMIC FUNCTIONS

WEEK 10

Logarithmic Properties and Equations

Logarithms are the inverse of exponentials — the question "what exponent do I need?" Learn log notation, the three power rules, and how to solve log equations.

Learning Objectives

By the end of this week, you will be able to:

1. Define a logarithm and its relationship to exponents.
2. Convert between exponential and logarithmic forms.
3. Apply logarithmic properties correctly.
4. Solve basic logarithmic equations.
5. Identify domain restrictions for logarithms.
6. Interpret graphs of logarithmic functions.
7. Apply logarithms to real-world problems.
8. Avoid common errors in log operations.

Key Concepts

1. What is a logarithm?

A logarithm answers the question: to what exponent must a base be raised to get a number? Log base b of x equals y means b to the y equals x.

2. Domain restriction: positive only

Logarithms are only defined for positive numbers. You cannot take the log of zero or any negative number.

3. The three log properties

Logs turn multiplication into addition, division into subtraction, and exponents into products. These three rules are the workhorses.

4. Converting and solving

To solve a log equation, convert it to exponential form. To solve a complex log equation, use the properties to combine multiple logs into one, then convert.

5. Worked examples

Five logarithm examples.

Worked Examples

Example 1. Find $\log_5(25)$.

Solution: $5^2 = 25$, so $\log_5(25) = 2$.

Example 2. Convert $4^3 = 64$ to log form.

Solution: $\log_4(64) = 3$.

Example 3. Find $\log_2(32)$.

Solution: $2^5 = 32$, so $\log_2(32) = 5$.

UNIT 3 · EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Week 10 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Find $\log_5(25)$.

2. Convert $4^3 = 64$ to log form.

3. Find $\log_2(32)$.

4. Simplify $\log(3) + \log(4)$.

5. Simplify $\log(100) - \log(4)$.

6. Expand $\log(x^2)$.

7. Solve $\log(x) = 3$.

8. Solve $\log_2(x) = 5$.

9. Solve $\ln(x) = 1$.

10. What is the domain of $f(x) = \log(x - 3)$?

11. What is the domain of $f(x) = \log(2x + 4)$?

12. Use change of base to express $\log_3(20)$.

13. Why are logarithms used on the Richter scale?

14. Find $\log_{10}(0.01)$.

15. Combine into a single log: $2 \log(x) + \log(y)$.

UNIT 3 · EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Week 10 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. $\log_2(8)$ = ?

(A) 4
(B) 8
(C) 3
(D) 2
Your answer: ____________

2. Convert $5^2 = 25$ to logarithmic form.

(A) $\log_5(25) = 2$
(B) $\log_{25}(5) = 2$
(C) $\log_5(2) = 25$
(D) $\log_2(25) = 5$
Your answer: ____________

3. Simplify log(4) + log(25).

(A) log(100) = 2
(B) log(29)
(C) 2 log(29)
(D) log(625)
Your answer: ____________

4. Expand $\log(x^5)$.

(A) log(5x)
(B) 5 log(x)
(C) log(x) / 5
(D) log(5) · log(x)
Your answer: ____________

5. Solve log(x) = 3.

(A) x = 10
(B) x = 3
(C) x = 1000
(D) x = 30
Your answer: ____________

6. What is the domain of f(x) = log(x)?

(A) $x \geq 0$
(B) All real numbers
(C) $x > 0$

(D) $x \neq 0$
Your answer: ____________

7. Simplify $\log(8) - \log(2)$.

(A) $\log(4)$
(B) $\log(16)$
(C) $\log(10)$
(D) $\log(6)$
Your answer: ____________

8. $\ln(e^5) = ?$

(A) 5e
(B) 1/5
(C) 5
(D) e^5
Your answer: ____________

DID YOU KNOW?

The Richter scale used to measure earthquakes is logarithmic: each whole number step represents a ten-fold increase in shaking amplitude and roughly 32 times more energy. So a magnitude 7 quake is not "a bit worse" than magnitude 6 — it releases about 32 times more energy. The pH scale, decibel scale for sound, and stellar magnitude scale for star brightness all work the same way.

UNIT 3 · EXPONENTIAL AND LOGARITHMIC FUNCTIONS

WEEK 11

Solving Exponential and Logarithmic Equations

Three methods, one principle: exponentials and logarithms are inverses. Master same-base matching, taking logs, and converting log equations — then learn to spot and reject extraneous solutions.

Learning Objectives

By the end of this week, you will be able to:

1. Solve exponential equations using common bases.
2. Solve exponential equations using logarithms.
3. Solve logarithmic equations algebraically.
4. Apply inverse relationships between logs and exponents.
5. Identify and reject extraneous solutions.
6. Interpret solutions graphically.
7. Apply exponential/log equations to real-world problems.
8. Demonstrate fluency in algebraic manipulation of logs and exponents.

Key Concepts

1. Method 1: Same base

If both sides of an exponential equation can be written with the same base, just match the exponents.

2. Method 2: Take logs of both sides

When bases cannot be matched, take the log of both sides and use the power rule to bring the variable down from the exponent.

3. Method 3: Solve log equations

To solve a logarithmic equation, convert it to exponential form. If multiple log terms appear, combine them first.

4. Watch for extraneous solutions

Always check your answers against the original equation. A solution that makes any log argument zero or negative must be discarded.

5. Worked examples

Five worked equations across the three methods.

Worked Examples

Example 1. Solve $2^x = 32$.

Solution: $32 = 2^5$, so $x = 5$.

Example 2. Solve $3^x = 27$.

Solution: $27 = 3^3$, so $x = 3$.

Example 3. Solve $5^x = 1/5$.

Solution: $1/5 = 5^{-1}$, so $x = -1$.

UNIT 3 · EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Week 11 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Solve $2^x = 32$.

2. Solve $3^x = 27$.

3. Solve $5^x = 1/5$.

4. Solve $2^x = 10$ using logs.

5. Solve $3^x = 50$.

6. Solve $\log(x) = 4$.

7. Solve $\ln(x) = 2$.

8. Solve $\log_3(x) = 4$.

9. Solve $\log(x - 1) = 2$.

10. Solve $\log(x) + \log(2) = \log(6)$.

11. Solve $\log(x) + \log(x - 3) = \log(10)$.

12. Solve $2^{x+1} = 16$.

13. Bacteria $P(t) = 500 \cdot 1.1^t$. When $P = 1500$?

14. Investment $A(t) = 1000(1.05)^t$. When does A double?

15. Drug decay $D(t) = 200 \cdot 0.5^{t/4}$. When $D = 25$?

UNIT 3 · EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Week 11 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. Solve $2^x = 32$.

- **(A)** $x = 32$
- **(B)** $x = 5$
- **(C)** $x = 16$
- **(D)** $x = 2$

Your answer: ____________

2. Solve $5^x = 125$.

- **(A)** $x = 125$
- **(B)** $x = 25$
- **(C)** $x = 3$
- **(D)** $x = 5$

Your answer: ____________

3. Express the solution to $3^x = 20$ using logs.

- **(A)** $x = \log(3)/\log(20)$
- **(B)** $x = \log(20)/\log(3)$
- **(C)** $x = \log(60)$
- **(D)** $x = \log(20) - \log(3)$

Your answer: ____________

4. Solve $\log(x) = 4$.

- **(A)** $x = 4$
- **(B)** $x = 10000$
- **(C)** $x = 40$
- **(D)** $x = 10$

Your answer: ____________

5. Solve $\log_2(x) = 5$.

- **(A)** $x = 5$
- **(B)** $x = 32$
- **(C)** $x = 25$
- **(D)** $x = 10$

Your answer: ____________

6. After combining $\log(x) + \log(x + 3) = \log(10)$, you get $x^2 + 3x - 10 = 0$. The solutions are 2 and -5. Which is extraneous?

- **(A)** $x = 2$
- **(B)** Both

(C) Neither
(D) $x = -5$
Your answer: ____________

7. Solve $\log(x - 1) = 2$.

(A) $x = 11$
(B) $x = 100$
(C) $x = 3$
(D) $x = 101$
Your answer: ____________

8. A bacteria culture $P(t) = 200 \cdot 1.5^t$. When does $P = 1000$? Express using logs.

(A) $t = \log(5)/\log(1.5)$
(B) $t = \log(1000)/\log(200)$
(C) $t = 5$
(D) $t = \log(1.5)/\log(5)$
Your answer: ____________

DID YOU KNOW?

The exact age of fossils, ancient bones, and even the Shroud of Turin is determined by an exponential equation: carbon-14 dating uses the half-life of radioactive carbon (5,730 years) to back-calculate when something was last alive. The mathematics here — solving $N(t) = N_0 \cdot e^{-kt}$ for t — is exactly what you are learning this week. Archaeology, geology, and forensic science all run on this single equation.

UNIT 3 · EXPONENTIAL AND LOGARITHMIC FUNCTIONS

WEEK 12

Applications to Real-Life Problems

The capstone of Unit 3. Choose the right function for the situation, build the model, solve, and interpret. Population growth, compound interest, decibels, pH, optimisation — every model from precalculus comes together.

Learning Objectives

By the end of this week, you will be able to:

1. Translate real-world scenarios into mathematical functions.
2. Select appropriate function types for given situations.
3. Interpret graphs in applied contexts.
4. Solve multi-step real-world problems.
5. Apply exponential, polynomial, and rational models.
6. Analyse and interpret results in context.
7. Evaluate whether a model is reasonable.
8. Communicate mathematical reasoning clearly.

Key Concepts

1. Choosing the right function

Different real-world situations call for different function types. Linear for constant rates, polynomial for area and profit, rational for ratios, exponential for growth and decay, logarithmic for scale measurements.

2. The modelling workflow

A seven-step process: identify quantities, define variables, choose function type, build the equation, solve, interpret in context, check reasonableness.

3. Common applications

Population growth, compound interest, radioactive decay, Newton's law of cooling, sound intensity, earthquake magnitude, and pH are the classic examples.

4. Interpreting your answer

A number alone is not an answer. The answer must include units, context, and a sanity check.

5. Worked examples

Five applied problems from across the function families.

Worked Examples

Example 1. £500 at 4% annual interest. Find value after 10 years.

Solution: $A = 500(1.04)^{10} \approx$ £740.12.

Example 2. A car travels 60 km/h for 4 hours. Distance?

Solution: Linear: $d = 60 \cdot 4 = 240$ km.

Example 3. $P(t) = 100(1.08)^t$. When does P reach 200?

Solution: $(1.08)^t = 2 \rightarrow t = \log(2)/\log(1.08) \approx 9$ years.

UNIT 3 · EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Week 12 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. £500 at 4% annual interest. Find value after 10 years.

2. A car travels 60 km/h for 4 hours. Distance?

3. $P(t) = 100(1.08)^t$. When does P reach 200?

4. A radioisotope has half-life 8 years. Starting with 1000 g, find amount after 24 years.

5. Sound is 100 times more intense than background. Decibels?

6. pH of a solution with $[H^+] = 0.001$ M?

7. Profit $P(x) = -x^2 + 12x - 20$. Find maximum.

8. Job rate R(x) = 80/x £/hr for x workers. Rate for 8 workers?

9. Bacteria culture $P(t) = 500(1.5)^t$. When P = 5000?

10. Magnitude 6 earthquake versus magnitude 4. Amplitude factor?

11. How long to triple money at 5% annual interest?

12. A drug half-life 6 hours. Starting 240 mg, when is amount 30 mg?

13. Continuous growth $P(t) = 1000\ e^{0.03t}$. P(20)?

14. Choose the model: text messages sent per day vs. number of people in a group.

15. A pH 4 solution vs. pH 7. Hydrogen ion concentration ratio?

UNIT 3 · EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Week 12 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. £1000 at 6% annual interest for 5 years. Approximate value?

- **(A)** £1338
- **(B)** £1300
- **(C)** £1600
- **(D)** £1060

Your answer: ____________

2. $P(t) = 200(1.05)^t$. Approximate P(10).

- **(A)** 500
- **(B)** 220
- **(C)** 326
- **(D)** 250

Your answer: ____________

3. A drug has half-life 4 hours. Starting with 800 mg, how much is left after 12 hours?

- **(A)** 100 mg
- **(B)** 50 mg
- **(C)** 200 mg
- **(D)** 400 mg

Your answer: ____________

4. For $P(x) = -x^2 + 10x - 16$, find maximum profit.

- **(A)** 9 at x = 5
- **(B)** 5 at x = 5
- **(C)** 16 at x = 4
- **(D)** 10 at x = 5

Your answer: ____________

5. A magnitude 8 earthquake is how many times the amplitude of a magnitude 5?

- **(A)** 3
- **(B)** 1000
- **(C)** 30
- **(D)** 100

Your answer: ____________

6. Which model best fits "money in a savings account with annual compounding"?

- **(A)** Polynomial
- **(B)** Exponential

(C) Rational
(D) Linear
Your answer: ____________

7. Which model best fits "fuel consumption per mile as distance varies"?

(A) Polynomial
(B) Rational
(C) Exponential
(D) Logarithmic
Your answer: ____________

8. $P(t) = 100e^{0.04t}$ describes a population. When does it double? (Rule of 70)

(A) About 50 years
(B) About 17.5 years
(C) About 4 years
(D) About 0.04 years
Your answer: ____________

DID YOU KNOW?

The 2008 financial crisis was partly driven by a misuse of mathematical models. Banks used the Gaussian copula formula to estimate the risk of bundled mortgages — but the formula assumed defaults were independent. When U.S. house prices fell, defaults turned out to be highly correlated, and the model collapsed. The lesson: a model is only as good as its assumptions, and reasonableness checks are not optional. The mathematics is easy; the modelling is hard.

ANSWER KEY

Unit 3 · Exponential and Logarithmic Functions

This answer key covers every practice exercise and quiz question from Unit 3. For full step-by-step solutions to randomised practice generators (separate from the worksheet exercises printed here), refer to the BemandaSTEM Precalculus app.

ANSWER KEY

Week 9 — Exponential Growth and Decay

Practice Exercises

1. **Q.** Evaluate $f(x) = 3 \cdot 2^x$ at $x = 3$.

   A. $f(3) = 3 \cdot 8 = 24$.

2. **Q.** Evaluate $f(x) = 100 \cdot 0.8^x$ at $x = 2$.

   A. $f(2) = 100 \cdot 0.64 = 64$.

3. **Q.** Is $f(x) = 5 \cdot 1.2^x$ growth or decay?

   A. Base $1.2 > 1$ — growth.

4. **Q.** Is $f(x) = 10 \cdot 0.6^x$ growth or decay?

   A. Base $0.6 < 1$ — decay.

5. **Q.** For $P(t) = 200 \cdot e^{0.05t}$, what is the initial population and the continuous rate?

   A. Initial 200; rate 5% per unit time.

6. **Q.** Estimate doubling time at 2% growth using Rule of 70.

   A. $70/2 = 35$ time units.

7. **Q.** Estimate doubling time at 10% growth.

   A. $70/10 = 7$ time units.

8. **Q.** A bacteria culture $P(t) = 500 \cdot 1.1^t$. Find $P(3)$.

   A. $P(3) = 500 \cdot 1.331 \approx 665.5$.

9. **Q.** A drug has half-life 4 hours. Starting with 200 mg, how much remains after 12 hours?

   A. $12/4 = 3$ half-lives: $200 \to 100 \to 50 \to 25$ mg.

10. **Q.** A radioisotope has half-life 5 years. After 20 years, what fraction remains?

    A. $20/5 = 4$ half-lives: $(1/2)^4 = 1/16$.

11. **Q.** For $f(x) = 2^x$, find $f(0)$, $f(1)$, $f(2)$, $f(3)$.

    A. 1, 2, 4, 8.

12. **Q.** For $f(x) = (1/2)^x$, find $f(0)$, $f(1)$, $f(2)$, $f(3)$.

    A. 1, 1/2, 1/4, 1/8.

13. **Q.** £1000 invested at 6% compounded annually. Write the model and find value after 5 years.

    A. $A(t) = 1000(1.06)^t$; $A(5) = 1000 \cdot 1.3382 \approx$ £1338.23.

14. **Q.** Which grows faster: $f(x) = 2^x$ or $g(x) = 1.5^x$?

    A. f, because base $2 > 1.5$.

15. **Q.** For $P(t) = 800 \cdot e^{-0.1t}$, is it growth or decay? Find $P(10)$.

    A. Decay ($k = -0.1$). $P(10) = 800 \cdot e^{-1} \approx 800 \cdot 0.368 \approx 294.3$.

Quiz Answers

1. **Answer: (A) $f(x) = 5 \cdot 2^x$**

   Reason: Base $2 > 1$, so this is exponential growth.

2. **Answer: (A) 100**

   Reason: $f(0) = 100 \cdot 0.8^0 = 100 \cdot 1 = 100$.

3. **Answer: (B) 2%**

   Reason: $k = 0.02$ corresponds to 2% per unit time.

4. **Answer: (D) 14 years**

   Reason: 70 / 5 = 14 years.

5. **Answer: (A) 1/8**

   Reason: 18 hours = 3 half-lives. After 3 halvings: $(1/2)^3 = 1/8$.

6. **Answer: (A) 2.718**

   Reason: Euler\\u2019s number $e \approx 2.71828$.

7. **Answer: (C) $0 < b < 1$**

   Reason: Decay requires a base between 0 and 1.

8. **Answer: (D) 48**

   Reason: $f(4) = 3 \cdot 2^4 = 3 \cdot 16 = 48$.

ANSWER KEY

Week 10 — Logarithmic Properties and Equations

Practice Exercises

1. **Q.** Find $\log_5(25)$.
   A. $5^2 = 25$, so $\log_5(25) = 2$.
2. **Q.** Convert $4^3 = 64$ to log form.
   A. $\log_4(64) = 3$.
3. **Q.** Find $\log_2(32)$.
   A. $2^5 = 32$, so $\log_2(32) = 5$.
4. **Q.** Simplify $\log(3) + \log(4)$.
   A. $\log(12)$.
5. **Q.** Simplify $\log(100) - \log(4)$.
   A. $\log(25)$.
6. **Q.** Expand $\log(x^2)$.
   A. $2 \log(x)$.
7. **Q.** Solve $\log(x) = 3$.
   A. $10^3 = x \rightarrow x = 1000$.
8. **Q.** Solve $\log_2(x) = 5$.
   A. $2^5 = x \rightarrow x = 32$.
9. **Q.** Solve $\ln(x) = 1$.
   A. $e^1 = x \rightarrow x = e \approx 2.718$.
10. **Q.** What is the domain of $f(x) = \log(x - 3)$?
    A. $x - 3 > 0$, so $x > 3$.
11. **Q.** What is the domain of $f(x) = \log(2x + 4)$?
    A. $2x + 4 > 0$, so $x > -2$.
12. **Q.** Use change of base to express $\log_3(20)$.
    A. $\log(20)/\log(3) \approx 2.727$.
13. **Q.** Why are logarithms used on the Richter scale?
    A. Earthquake amplitudes span many orders of magnitude; a log scale compresses them so each whole number = 10× the previous.
14. **Q.** Find $\log_{10}(0.01)$.
    A. $10^{-2} = 0.01$, so $\log(0.01) = -2$.
15. **Q.** Combine into a single log: $2 \log(x) + \log(y)$.
    A. $\log(x^2) + \log(y) = \log(x^2 \cdot y)$.

Quiz Answers

1. **Answer: (C) 3**
   Reason: $2^3 = 8$, so $\log_2(8) = 3$.
2. **Answer: (A) $\log_5(25) = 2$**

Reason: Pattern $b^y = x \Leftrightarrow \log_b(x) = y$, so $\log_5(25) = 2$.

3. **Answer: (A) log(100) = 2**

   Reason: Product rule: $\log(4 \cdot 25) = \log(100)$. And $\log(100) = 2$.

4. **Answer: (B) 5 log(x)**

   Reason: Power rule: $\log(x^n) = n \log(x)$, so $\log(x^5) = 5 \log(x)$.

5. **Answer: (C) x = 1000**

   Reason: Convert: $10^3 = x$, so $x = 1000$.

6. **Answer: (C) x > 0**

   Reason: Logs are only defined for strictly positive arguments.

7. **Answer: (A) log(4)**

   Reason: Quotient rule: $\log(8/2) = \log(4)$.

8. **Answer: (C) 5**

   Reason: ln and e are inverses, so $\ln(e^5) = 5$.

ANSWER KEY

Week 11 — Solving Exponential and Logarithmic Equations

Practice Exercises

1. **Q.** Solve $2^x = 32$.
   A. $32 = 2^5$, so $x = 5$.
2. **Q.** Solve $3^x = 27$.
   A. $27 = 3^3$, so $x = 3$.
3. **Q.** Solve $5^x = 1/5$.
   A. $1/5 = 5^{-1}$, so $x = -1$.
4. **Q.** Solve $2^x = 10$ using logs.
   A. $x = \log(10)/\log(2) \approx 3.32$.
5. **Q.** Solve $3^x = 50$.
   A. $x = \log(50)/\log(3) \approx 3.56$.
6. **Q.** Solve $\log(x) = 4$.
   A. $10^4 = x \to x = 10000$.
7. **Q.** Solve $\ln(x) = 2$.
   A. $e^2 \approx 7.39$, so $x \approx 7.39$.
8. **Q.** Solve $\log_3(x) = 4$.
   A. $3^4 = x = 81$.
9. **Q.** Solve $\log(x - 1) = 2$.
   A. $x - 1 = 100 \to x = 101$.
10. **Q.** Solve $\log(x) + \log(2) = \log(6)$.
    A. $\log(2x) = \log(6) \to 2x = 6 \to x = 3$.
11. **Q.** Solve $\log(x) + \log(x - 3) = \log(10)$.
    A. $\log(x^2 - 3x) = \log(10) \to x^2 - 3x - 10 = 0 \to (x - 5)(x + 2) = 0$. Check: $x > 3$ needed, so $x = 5$.
12. **Q.** Solve $2^{x+1} = 16$.
    A. $16 = 2^4$, so $x + 1 = 4 \to x = 3$.
13. **Q.** Bacteria $P(t) = 500 \cdot 1.1^t$. When $P = 1500$?
    A. $1.1^t = 3 \to t = \log(3)/\log(1.1) \approx 11.5$.
14. **Q.** Investment $A(t) = 1000(1.05)^t$. When does A double?
    A. $1.05^t = 2 \to t = \log(2)/\log(1.05) \approx 14.2$ years.
15. **Q.** Drug decay $D(t) = 200 \cdot 0.5^{t/4}$. When $D = 25$?
    A. $0.5^{t/4} = 1/8 = 0.5^3$, so $t/4 = 3$, $t = 12$ hours.

Quiz Answers

1. **Answer: (B) $x = 5$**
   Reason: $32 = 2^5$, so $x = 5$.
2. **Answer: (C) $x = 3$**

Reason: $125 = 5^3$, so $x = 3$.

3. **Answer: (B)** $\mathbf{x = log(20)/log(3)}$

   Reason: Take log of both sides: $x \cdot \log(3) = \log(20)$, so $x = \log(20)/\log(3)$.

4. **Answer: (B)** $\mathbf{x = 10000}$

   Reason: Convert: $10^4 = x = 10000$.

5. **Answer: (B)** $\mathbf{x = 32}$

   Reason: $2^5 = x = 32$.

6. **Answer: (D)** $\mathbf{x = -5}$

   Reason: $\log(x)$ needs $x > 0$, so $x = -5$ fails the domain check. Reject it; $x = 2$ is the valid answer.

7. **Answer: (D)** $\mathbf{x = 101}$

   Reason: Convert: $10^2 = x - 1 \rightarrow x - 1 = 100 \rightarrow x = 101$.

8. **Answer: (A)** $\mathbf{t = log(5)/log(1.5)}$

   Reason: $1000 = 200 \cdot 1.5^t \rightarrow 1.5^t = 5 \rightarrow t = \log(5)/\log(1.5)$.

ANSWER KEY

Week 12 — Applications to Real-Life Problems

Practice Exercises

1. **Q.** £500 at 4% annual interest. Find value after 10 years.
   A. $A = 500(1.04)^{10} \approx$ £740.12.
2. **Q.** A car travels 60 km/h for 4 hours. Distance?
   A. Linear: $d = 60 \cdot 4 = 240$ km.
3. **Q.** $P(t) = 100(1.08)^t$. When does P reach 200?
   A. $(1.08)^t = 2 \rightarrow t = \log(2)/\log(1.08) \approx 9$ years.
4. **Q.** A radioisotope has half-life 8 years. Starting with 1000 g, find amount after 24 years.
   A. $24/8 = 3$ half-lives: $1000 \rightarrow 500 \rightarrow 250 \rightarrow 125$ g.
5. **Q.** Sound is 100 times more intense than background. Decibels?
   A. $L = 10 \log(100) = 10 \cdot 2 = 20$ dB.
6. **Q.** pH of a solution with $[H^+] = 0.001$ M?
   A. $pH = -\log(0.001) = -(-3) = 3$.
7. **Q.** Profit $P(x) = -x^2 + 12x - 20$. Find maximum.
   A. Vertex $x = 6$, $P(6) = -36 + 72 - 20 = 16$. Max £16.
8. **Q.** Job rate $R(x) = 80/x$ £/hr for x workers. Rate for 8 workers?
   A. $R(8) = 10$ £/hr each.
9. **Q.** Bacteria culture $P(t) = 500(1.5)^t$. When $P = 5000$?
   A. $(1.5)^t = 10 \rightarrow t = \log(10)/\log(1.5) \approx 5.68$ hours.
10. **Q.** Magnitude 6 earthquake versus magnitude 4. Amplitude factor?
    A. $10^2 = 100\times$ larger.
11. **Q.** How long to triple money at 5% annual interest?
    A. $(1.05)^t = 3 \rightarrow t = \log(3)/\log(1.05) \approx 22.5$ years.
12. **Q.** A drug half-life 6 hours. Starting 240 mg, when is amount 30 mg?
    A. $240/30 = 8 = 2^3$, so 3 half-lives = 18 hours.
13. **Q.** Continuous growth $P(t) = 1000\, e^{0.03t}$. P(20)?
    A. $1000\, e^{0.6} \approx 1000 \cdot 1.822 \approx 1822$.
14. **Q.** Choose the model: text messages sent per day vs. number of people in a group.
    A. Polynomial — number of pairs (and so messages) grows roughly as $n(n-1)/2$.
15. **Q.** A pH 4 solution vs. pH 7. Hydrogen ion concentration ratio?
    A. $10^{7-4} = 1000\times$ more H^+ in the pH 4 solution.

Quiz Answers

1. **Answer: (A) £1338**
   Reason: $A = 1000(1.06)^5 \approx$ £1338.
2. **Answer: (C) 326**
   Reason: $P(10) = 200 \cdot 1.629 \approx 326$.

3. **Answer: (A) 100 mg**

   Reason: 12/4 = 3 half-lives: 800 → 400 → 200 → 100 mg.

4. **Answer: (A) 9 at x = 5**

   Reason: Vertex $x = 5$, $P(5) = -25 + 50 - 16 = 9$.

5. **Answer: (B) 1000**

   Reason: Difference of 3 magnitudes = $10^3 = 1000\times$ the amplitude.

6. **Answer: (B) Exponential**

   Reason: Compound interest is exponential: $A = P(1 + r)^t$.

7. **Answer: (B) Rational**

   Reason: Rate = consumption/distance is a rational model.

8. **Answer: (B) About 17.5 years**

   Reason: Rule of 70: 70 / 4 = 17.5 years.

UNIT 4
Trigonometry

UNIT OVERVIEW

The biggest unit in the course — eight weeks of triangles, angles, and circular motion. Right-triangle ratios, oblique-triangle laws, the unit circle, the graphs of sin/cos/tan, identities, equations, and a final week applying it all to sound, light, tides, and engineering.

Weeks in this unit:

Week 13 — *Trigonometric Ratios*
Week 14 — *Right Triangle Problems*
Week 15 — *Oblique Triangles*
Week 16 — *Unit Circle Basics*
Week 17 — *Graphing Trigonometric Functions*
Week 18 — *Trigonometric Identities*
Week 19 — *Solving Trigonometric Equations*
Week 20 — *Applications & Unit Review*

UNIT 4 · TRIGONOMETRY

WEEK 13

Trigonometric Ratios

Meet sine, cosine, and tangent — the three ratios that connect the angles of a right triangle to the lengths of its sides. SOH-CAH-TOA, inverse functions, and angles of elevation.

Learning Objectives

By the end of this week, you will be able to:

1. Define trigonometric ratios in right triangles.
2. Use SOH-CAH-TOA to identify ratios.
3. Calculate sine, cosine, and tangent values.
4. Solve for missing sides in right triangles.
5. Solve for missing angles using inverse trig ratios.
6. Apply trig ratios to real-world problems.
7. Interpret angle of elevation and depression.
8. Use calculators correctly for trig functions.

Key Concepts

1. The three ratios — SOH-CAH-TOA

Every right triangle has three trigonometric ratios. SOH-CAH-TOA is the mnemonic: sine equals opposite over hypotenuse, cosine equals adjacent over hypotenuse, tangent equals opposite over adjacent.

2. Labelling the sides

In a right triangle, the hypotenuse is the longest side, opposite the right angle. The opposite side is across from your reference angle, and the adjacent side is next to it.

3. Inverse trig: finding angles

When you know two sides but not the angle, use inverse trig functions: arcsin, arccos, and arctan — written $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$.

4. Angles of elevation and depression

The angle of elevation looks up from horizontal. The angle of depression looks down from horizontal. Both are measured from a horizontal sight line.

5. Worked examples

Five worked trigonometry examples covering ratios, finding sides, and finding angles.

Worked Examples

Example 1. Find $\sin(\theta)$ when opposite = 8 and hypotenuse = 10.

Solution: $8/10 = 4/5 = 0.8$.

Example 2. Find $\cos(\theta)$ when adjacent = 5 and hypotenuse = 13.

Solution: $5/13 \approx 0.385$.

Example 3. Find $\tan(\theta)$ when opposite = 9 and adjacent = 12.

Solution: $9/12 = 3/4 = 0.75$.

UNIT 4 · TRIGONOMETRY

Week 13 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Find $\sin(\theta)$ when opposite = 8 and hypotenuse = 10.

2. Find $\cos(\theta)$ when adjacent = 5 and hypotenuse = 13.

3. Find $\tan(\theta)$ when opposite = 9 and adjacent = 12.

4. If $\sin(\theta) = 0.6$ and hypotenuse = 20, find the opposite.

5. If $\cos(\theta) = 0.8$ and hypotenuse = 25, find the adjacent.

6. Find θ if $\tan(\theta) = 2$.

7. Find θ if $\sin(\theta) = 0.5$.

8. A ladder is leaning against a wall at 60° angle with the ground. The ladder is 10 m long. Find the height reached.

9. A ramp rises 2 m over a horizontal distance of 8 m. Find the angle of elevation.

10. A tower is observed from 50 m away. Angle of elevation to the top is 30°. Find the tower\\u2019s height.

11. From the top of a 40 m cliff, a boat is observed at 25° angle of depression. Find the horizontal distance.

12. Find sin(θ) where opposite = 3 and adjacent = 4.

13. A kite is on a 50 m string at 70° angle from the ground. Height?

14. For sin(θ) = 0.5, cos(θ) = 0.5: is this possible?

15. A 20 m flagpole casts a 12 m shadow. Find the sun\\u2019s angle of elevation.

UNIT 4 · TRIGONOMETRY

Week 13 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. In a right triangle, opposite = 5, hypotenuse = 13. Find sin(θ).

- **(A)** 12/13
- **(B)** 5/12
- **(C)** 13/5
- **(D)** 5/13

Your answer: ____________

2. In a right triangle, adjacent = 8, hypotenuse = 17. Find cos(θ).

- **(A)** 17/8
- **(B)** 15/17
- **(C)** 8/15
- **(D)** 8/17

Your answer: ____________

3. In a right triangle, opposite = 7, adjacent = 24. Find tan(θ).

- **(A)** 7/25
- **(B)** 25/7
- **(C)** 24/7
- **(D)** 7/24

Your answer: ____________

4. If sin(θ) = 0.5 and hypotenuse = 20, find the opposite side.

- **(A)** 5
- **(B)** 20
- **(C)** 10
- **(D)** 40

Your answer: ____________

5. If tan(θ) = 1, what is θ (for $0° \le \theta \le 90°$)?

- **(A)** 60°
- **(B)** 90°
- **(C)** 45°
- **(D)** 30°

Your answer: ____________

6. A ladder 10 m long makes a 60° angle with the ground. Height reached?

- **(A)** 8.66 m
- **(B)** 7.07 m
- **(C)** 5 m

(D) 10 m

Your answer: ____________

7. Which side is "opposite" the angle θ in a right triangle?

(A) The side under θ
(B) The longest side
(C) The side across from θ
(D) The side next to θ

Your answer: ____________

8. $\sin^{-1}(0.5) = ?$

(A) 45°
(B) 60°
(C) 0°
(D) 30°

Your answer: ____________

DID YOU KNOW?

Trigonometry is one of the oldest branches of mathematics — the Babylonians built tables of trig ratios over 3,500 years ago to predict the motion of stars and planets. The Greek astronomer Hipparchus, working around 150 BCE, compiled the first complete trig tables. Without those tables, the entire history of astronomy, navigation, and architecture (from the pyramids to GPS satellites) would have been impossible.

UNIT 4 · TRIGONOMETRY

WEEK 14

Right Triangle Problems

Now apply SOH-CAH-TOA at scale. Combine trig ratios with the Pythagorean theorem to solve full right triangles, then tackle multi-step real-world problems: shadows, ramps, ladders, towers, lines of sight.

Learning Objectives

By the end of this week, you will be able to:

1. Solve right triangle problems using trig ratios.
2. Apply the Pythagorean Theorem when needed.
3. Determine missing sides and angles.
4. Select appropriate trigonometric ratios.
5. Solve multi-step word problems.
6. Interpret angle of elevation and depression.
7. Use calculators accurately in calculations.
8. Check solutions for reasonableness.

Key Concepts

1. The Pythagorean theorem

In any right triangle, the sum of the squares of the two legs equals the square of the hypotenuse: a squared plus b squared equals c squared.

2. Choose your method

When all three sides matter, use Pythagorean. When an angle is involved, use trigonometry. The right tool depends on what you know and what you need.

3. Five-step solving strategy

Draw and label the triangle, identify what is known, choose the method, solve step by step, check the answer for reasonableness.

4. Common application templates

A ladder against a wall, a ramp rising at an angle, a flagpole and its shadow, a sight line from a tower — most word problems map onto one of these templates.

5. Worked examples

Five worked right-triangle problems.

Worked Examples

Example 1. Find the hypotenuse of a right triangle with legs 9 and 12.

Solution: $c = \sqrt{(81 + 144)} = \sqrt{225} = 15$.

Example 2. Find the missing leg if hypotenuse = 17 and one leg = 8.

Solution: $b = \sqrt{(289 - 64)} = \sqrt{225} = 15$. (8-15-17 triple.)

Example 3. Find the missing leg if hypotenuse = 25 and one leg = 7.

Solution: $b = \sqrt{(625 - 49)} = \sqrt{576} = 24$. (7-24-25 triple.)

UNIT 4 · TRIGONOMETRY

Week 14 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Find the hypotenuse of a right triangle with legs 9 and 12.

2. Find the missing leg if hypotenuse = 17 and one leg = 8.

3. Find the missing leg if hypotenuse = 25 and one leg = 7.

4. $\theta = 45°$, hypotenuse = 14. Find both legs.

5. $\theta = 30°$, hypotenuse = 20. Find the opposite leg.

6. $\theta = 60°$, hypotenuse = 12. Find the adjacent leg.

7. A 50 m tower is observed from 30° elevation. Find the distance from base.

8. A 5 m ramp at 20° angle. Find the height.

9. From the top of a 30 m cliff, a boat is seen at 25° depression. Distance to boat?

10. sin(θ) = 5/13. Find tan(θ) using Pythagorean.

11. A pole 8 m tall casts a 6 m shadow. Find the sun\\u2019s elevation.

12. A 10 m ladder reaches 8 m up a wall. Distance from wall?

13. Find the angle of a 5-12-13 right triangle opposite the side of length 5.

14. A right triangle has angles 30°, 60°, 90°. If the shortest side is 4, find the other sides.

15. A flagpole leans 5° from vertical and casts a shadow at noon. Is this a right triangle problem? Why or why not?

UNIT 4 · TRIGONOMETRY

Week 14 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. In a right triangle with legs 6 and 8, find the hypotenuse.

- **(A)** 7
- **(B)** 10
- **(C)** 12
- **(D)** 14

Your answer: ____________

2. A right triangle has hypotenuse 13 and one leg 5. Find the other leg.

- **(A)** 18
- **(B)** 7
- **(C)** 8
- **(D)** 12

Your answer: ____________

3. $\theta = 45°$, hypotenuse = 10. Find the opposite side.

- **(A)** $10 \cdot \sqrt{2}/2 \approx 7.07$
- **(B)** 10
- **(C)** 5
- **(D)** 14.14

Your answer: ____________

4. A 15 m ladder leans against a wall at 60°. How high up the wall?

- **(A)** 7.5 m
- **(B)** 10.39 m
- **(C)** ≈ 12.99 m
- **(D)** 15 m

Your answer: ____________

5. $\tan(\theta) = 3/4$. Find θ.

- **(A)** ≈ 36.87°
- **(B)** 60°
- **(C)** 30°
- **(D)** 45°

Your answer: ____________

6. A building has a 20 m shadow when the sun is at 45° elevation. Building height?

- **(A)** 20 m
- **(B)** 10 m
- **(C)** 40 m

(D) 14.14 m

Your answer: ____________

7. A ramp rises 3 m over 4 m horizontal. Find the angle.

(A) 60°
(B) 45°
(C) 30°
(D) ≈ 36.87°

Your answer: ____________

8. Which method finds a side when you know two other sides only?

(A) Pythagorean theorem
(B) Law of Sines
(C) Inverse trig
(D) SOH-CAH-TOA

Your answer: ____________

DID YOU KNOW?

The ancient Egyptians used the 3-4-5 right triangle long before Pythagoras formalised the theorem. "Rope stretchers" carried lengths of rope with knots tied at intervals of 3, 4, and 5 units. Pulled tight, the rope formed a perfect right triangle, which they used to mark out the corners of the pyramids and resurvey the Nile floodplain every year after the flood. Pythagoras lived around 570 BCE; the rope stretchers were doing this 2,000 years earlier.

UNIT 4 · TRIGONOMETRY

WEEK 15

Oblique Triangles

When there is no right angle, SOH-CAH-TOA fails. Enter the Law of Sines and Law of Cosines — the universal triangle-solving tools used by surveyors, navigators, and astronomers for centuries.

Learning Objectives

By the end of this week, you will be able to:

1. Define an oblique triangle.
2. Apply the Law of Sines to solve triangles.
3. Apply the Law of Cosines to solve triangles.
4. Determine which method to use for given problems.
5. Solve SSS, SAS, ASA, and AAS triangles.
6. Recognise and resolve ambiguous SSA cases.
7. Calculate area of oblique triangles.
8. Apply oblique triangle concepts to real-world problems.

Key Concepts

1. Beyond the right angle

An oblique triangle has no right angle. All three angles are acute, or one is obtuse. SOH-CAH-TOA does not work directly — we need new tools.

2. The Law of Sines

For any triangle, the ratio of each side to the sine of its opposite angle is the same constant. Sine A over a equals sine B over b equals sine C over c.

3. The Law of Cosines

The Law of Cosines generalises the Pythagorean theorem. C squared equals a squared plus b squared minus 2 a b cosine of the angle between them.

4. Method selection and the ambiguous case

Use Law of Sines for ASA and AAS, Law of Cosines for SSS and SAS. The SSA case can produce zero, one, or two valid triangles.

5. Area and worked examples

The area of any triangle is one half a times b times sine of C, where C is the included angle.

Worked Examples

Example 1. $A = 40°$, $B = 70°$, $a = 15$. Find b.

Solution: $b = 15 \sin(70°)/\sin(40°) \approx 21.93$.

Example 2. $A = 50°$, $b = 12$, $c = 18$. Find a (Law of Cosines).

Solution: $a^2 = 144 + 324 - 432 \cos(50°) \approx 190.36$. $a \approx 13.80$.

Example 3. $a = 7$, $b = 9$, $C = 60°$. Find c.

Solution: $c^2 = 49 + 81 - 126 \cdot 0.5 = 67$. $c \approx 8.19$.

UNIT 4 · TRIGONOMETRY

Week 15 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. $A = 40°$, $B = 70°$, $a = 15$. Find b.

2. $A = 50°$, $b = 12$, $c = 18$. Find a (Law of Cosines).

3. $a = 7$, $b = 9$, $C = 60°$. Find c.

4. $a = 9$, $b = 10$, $c = 12$. Find angle A.

5. Find area with $a = 7$, $b = 9$, $C = 45°$.

6. Find area with $a = 10$, $b = 8$, $C = 60°$.

7. A surveyor measures two sides of a triangular plot: 50 m and 80 m with included angle 70°. Find the third side.

8. Two ships travel from port; one 25 km north, the other 40 km at 60° east of north. Distance between?

9. A triangle has A = 35°, a = 12, b = 18. Find B (Law of Sines).

10. In a triangle, A = 60°, B = 80°, c = 14. Find a.

11. Find C if cos(C) = 0.5.

12. Two sides 12 and 18, area = 50 square units. Find the included angle.

13. If A = 30°, B = 60°, what type of triangle is it (angle-wise)?

14. For SSA case with A = 30°, a = 5, b = 12, why is it problematic?

15. Find the area of a triangle with sides 6, 8, and included angle $90°$.

UNIT 4 · TRIGONOMETRY

Week 15 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. Which case requires the Law of Cosines?

- **(A)** None
- **(B)** ASA (two angles and included side)
- **(C)** SAS (two sides and included angle)
- **(D)** AAS (two angles and a non-included side)

Your answer: ____________

2. In a triangle, A = 30°, B = 60°, a = 10. Find b.

- **(A)** 10
- **(B)** ≈ 8.66
- **(C)** 20
- **(D)** ≈ 17.32

Your answer: ____________

3. In a triangle, a = 5, b = 7, C = 60°. Find c.

- **(A)** 2
- **(B)** ≈ 6.24
- **(C)** 12
- **(D)** ≈ 8.66

Your answer: ____________

4. In a triangle with a = 10, b = 12, included angle C = 30°, the area is...

- **(A)** 60
- **(B)** 15
- **(C)** 120
- **(D)** 30

Your answer: ____________

5. If A = 50° and B = 70°, what is C?

- **(A)** 120°
- **(B)** 180°
- **(C)** 60°
- **(D)** 100°

Your answer: ____________

6. Two ships leave port: 30 km and 40 km, with 90° between courses. Distance apart?

- **(A)** 70 km
- **(B)** 60 km

(C) 10 km
(D) 50 km
Your answer: ____________

7. For a = 9, b = 10, c = 12 (SSS), which law finds angle A?

(A) Law of Sines
(B) Pythagorean
(C) Law of Cosines
(D) SOH-CAH-TOA
Your answer: ____________

8. What is special about SSA cases?

(A) They require the Pythagorean theorem
(B) They can have 0, 1, or 2 triangles (ambiguous)
(C) They always have exactly one solution
(D) They cannot be solved
Your answer: ____________

DID YOU KNOW?

The Law of Cosines was originally proved by the Persian mathematician al-Kashi around 1427, and is still called the "al-Kashi theorem" in France. Long before GPS, navigators on ships used the Law of Sines combined with celestial observations to fix their position to within a few kilometres in the middle of an ocean. Surveyors mapping the Indian subcontinent in the 1800s used these same laws to measure the height of Mount Everest to within 9 metres — without ever touching the mountain.

UNIT 4 · TRIGONOMETRY

WEEK 16

Unit Circle Basics

The unit circle is the bridge from triangle trig to function trig. Every angle becomes a point (cos θ, sin θ) on a circle of radius 1 — and that unlocks trig for any angle, not just acute ones.

Learning Objectives

By the end of this week, you will be able to:

1. Define the unit circle and explain its purpose.
2. Convert between degrees and radians for key angles.
3. Identify coordinates of special angles on the unit circle.
4. Use the unit circle to determine sine and cosine values.
5. Apply quadrant sign rules (ASTC) correctly.
6. Determine reference angles for basic cases.
7. Interpret trigonometric values geometrically.
8. Apply unit circle concepts to basic real-world contexts.

Key Concepts

1. What is the unit circle?

The unit circle is a circle of radius 1 centered at the origin. For any angle theta measured from the positive x-axis, the point on the circle has coordinates cosine theta and sine theta.

2. Radians and degrees

A full circle is 360 degrees or 2 pi radians. Use radians when working with calculus or natural function behaviour. Use degrees for measuring angles in everyday situations.

3. Special angle coordinates

Five angles to memorise — 0, 30, 45, 60, and 90 degrees — and their coordinates on the unit circle. Everything else builds from these.

4. ASTC — quadrant signs

ASTC tells you which trig functions are positive in each quadrant. All in I, Sine in II, Tangent in III, Cosine in IV. The mnemonic: All Students Take Calculus.

5. Worked examples

Five worked unit-circle examples.

Worked Examples

Example 1. Find coordinates of 30°.

Solution: $(\sqrt{3}/2, 1/2)$.

Example 2. Find coordinates of 45°.

Solution: $(\sqrt{2}/2, \sqrt{2}/2)$.

Example 3. Find coordinates of 90°.

Solution: $(0, 1)$.

UNIT 4 · TRIGONOMETRY

Week 16 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Find coordinates of 30°.

2. Find coordinates of 45°.

3. Find coordinates of 90°.

4. Convert 180° to radians.

5. Convert 60° to radians.

6. Convert $\pi/2$ to degrees.

7. Convert $5\pi/6$ to degrees.

8. In which quadrant is 120°? What signs do sin, cos, tan have?

9. In which quadrant is 225°? Signs?

10. Find the reference angle for 135°.

11. Find the reference angle for 300°.

12. Find sin(150°).

13. Find cos(240°).

14. Find tan(315°).

15. Verify $\sin^2(30°) + \cos^2(30°) = 1$.

UNIT 4 · TRIGONOMETRY

Week 16 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. What are the coordinates of 60° on the unit circle?

(A) $(1/2, \sqrt{3}/2)$
(B) $(\sqrt{2}/2, \sqrt{2}/2)$
(C) $(\sqrt{3}/2, 1/2)$
(D) (0, 1)
Your answer: ____________

2. Convert 90° to radians.

(A) $\pi/4$
(B) π
(C) $\pi/3$
(D) $\pi/2$
Your answer: ____________

3. Convert $\pi/3$ radians to degrees.

(A) 90°
(B) 30°
(C) 60°
(D) 45°
Your answer: ____________

4. In which quadrant is 210°?

(A) III
(B) II
(C) IV
(D) I
Your answer: ____________

5. At 225°, the sign of cosine is…

(A) negative
(B) undefined
(C) zero
(D) positive
Your answer: ____________

6. Reference angle of 150°?

(A) 90°
(B) 30°
(C) 60°

(D) 45°
Your answer: ____________

7. sin(180°) = ?

(A) −1
(B) 0
(C) undefined
(D) 1
Your answer: ____________

8. cos(270°) = ?

(A) 1
(B) −1
(C) 0
(D) undefined
Your answer: ____________

DID YOU KNOW?

The unit circle has been called "the most beautiful diagram in mathematics" because so much converges in it. Sine and cosine are not just numbers but actual coordinates. The Pythagorean theorem becomes the identity $\sin^2 + \cos^2 = 1$ (since $x^2 + y^2 = 1$ on the unit circle). And the periodicity of every wave in physics — sound, light, water, electromagnetic — emerges from the simple fact that going around a circle, you eventually return to where you started.

UNIT 4 · TRIGONOMETRY

WEEK 17

Graphing Trigonometric Functions

Sine, cosine, and tangent as functions of x — waves that ripple forever. Master amplitude, period, midline, and phase shift, the four knobs that control any trig graph.

Learning Objectives

By the end of this week, you will be able to:

1. Graph basic sine, cosine, and tangent functions.
2. Identify amplitude, period, and midline.
3. Apply transformations to trig functions.
4. Recognise phase shifts in graphs.
5. Compare sine and cosine behaviour.
6. Interpret key points on trig graphs.
7. Model periodic real-world situations.
8. Analyse and describe trig graphs accurately.

Key Concepts

1. Sine and cosine as functions

When you plot the y-coordinate of a point on the unit circle against the angle x, you get the sine function. Plot the x-coordinate and you get cosine. Both produce smooth waves that repeat every 2π.

2. Amplitude, period, midline

Amplitude controls height. Period controls width. Midline is the horizontal centre line. Three knobs that shape any sinusoidal graph.

3. The general form

The full sinusoidal form is y equals a sine of b times x minus c, plus d. Each letter controls one feature.

4. Modelling periodic phenomena

Sound waves, tides, daylight hours, Ferris wheel heights, alternating current — anything periodic can be modelled by a sinusoid.

5. Worked examples

Five examples of reading trig graphs.

Worked Examples

Example 1. Find amplitude of $y = 2\sin(x)$.

Solution: Amplitude = 2.

Example 2. Find period of $y = \sin(4x)$.

Solution: $2\pi/4 = \pi/2$.

Example 3. Find midline of $y = \sin(x) + 5$.

Solution: $y = 5$.

UNIT 4 · TRIGONOMETRY

Week 17 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Find amplitude of $y = 2\sin(x)$.

2. Find period of $y = \sin(4x)$.

3. Find midline of $y = \sin(x) + 5$.

4. For $y = 3\cos(2x) - 1$, give amp, period, midline.

5. Find max of $y = 2\sin(x) + 3$.

6. Find min of $y = 2\sin(x) + 3$.

7. $\sin(0) = ?$

8. cos(0) = ?

9. sin(π) = ?

10. A tide goes from 0.5 m to 4.5 m over 12 hours. Model the height.

11. Compare y = sin(x) and y = cos(x). How are they related?

12. Find period of y = tan(2x).

13. Is y = sin(x) an even or odd function?

14. Is y = cos(x) an even or odd function?

15. A piston moves up and down 30 cm peak-to-peak, 60 cycles per second. Amplitude and period?

UNIT 4 · TRIGONOMETRY

Week 17 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. Amplitude of $y = 4 \sin(x)$?

- **(A)** 2
- **(B)** 1
- **(C)** 8
- **(D)** 4

Your answer: ____________

2. Period of $y = \sin(3x)$?

- **(A)** 3π
- **(B)** 2π
- **(C)** $\pi/3$
- **(D)** $2\pi/3$

Your answer: ____________

3. Midline of $y = \cos(x) - 2$?

- **(A)** $x = -2$
- **(B)** $y = 0$
- **(C)** $y = -2$
- **(D)** $y = 2$

Your answer: ____________

4. Maximum of $y = 5 \sin(x) + 1$?

- **(A)** 4
- **(B)** 6
- **(C)** 5
- **(D)** 1

Your answer: ____________

5. $\sin(\pi/2) = ?$

- **(A)** -1
- **(B)** 1
- **(C)** $\pi/2$
- **(D)** 0

Your answer: ____________

6. $\cos(\pi) = ?$

- **(A)** 0
- **(B)** π
- **(C)** -1

(D) 1

Your answer: ____________

7. Period of y = tan(x)?

(A) π/2
(B) 2π
(C) 3π
(D) π

Your answer: ____________

8. A Ferris wheel goes 2 m to 22 m. Amplitude is…

(A) 22 m
(B) 12 m
(C) 20 m
(D) 10 m

Your answer: ____________

DID YOU KNOW?

Joseph Fourier proved in 1822 that any periodic function — no matter how complex — can be decomposed into a sum of sines and cosines. This is the Fourier transform, and it powers MP3 audio compression, JPEG images, MRI scans, telecommunications, and noise-cancelling headphones. Every wave you have ever heard or seen on a screen has been processed using sums of sines.

UNIT 4 · TRIGONOMETRY

WEEK 18

Trigonometric Identities

Identities are equations that are true for every value of the variable. Master the Pythagorean, reciprocal, quotient, and even/odd identities — the algebra of trig.

Learning Objectives

By the end of this week, you will be able to:

1. Define a trigonometric identity.
2. State and use fundamental trig identities.
3. Simplify trigonometric expressions using identities.
4. Apply Pythagorean identities correctly.
5. Use reciprocal and quotient identities.
6. Identify even and odd trig functions.
7. Verify simple identities algebraically.
8. Apply trig identities in problem-solving contexts.

Key Concepts

1. Identity vs equation

An identity is an equation true for every value of the variable. An equation has solutions — specific values where both sides match.

2. Pythagorean identities

The three Pythagorean identities all stem from the unit circle: sine squared plus cosine squared equals one.

3. Reciprocal and quotient identities

Six trig functions come in three reciprocal pairs. Tangent and cotangent are quotients of sine and cosine.

4. Even and odd identities

Cosine is even — cosine of negative theta equals cosine of theta. Sine and tangent are odd — flipping the sign of theta flips the function value.

5. Worked examples

Five worked identity simplifications.

Worked Examples

Example 1. Simplify $1 - \cos^2\theta$.

Solution: $\sin^2\theta$.

Example 2. Simplify $\sin\theta / \cos\theta$.

Solution: $\tan\theta$.

Example 3. Express $\sec\theta$ in basic form.

Solution: $1/\cos\theta$.

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Week 18 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Simplify $1 - \cos^2\theta$.

2. Simplify $\sin\theta / \cos\theta$.

3. Express $\sec\theta$ in basic form.

4. Express $\cot\theta$ in basic form.

5. Verify $\sin\theta / \cos\theta = \tan\theta$.

6. Find $\sin(-45°)$.

7. Find $\tan(-60°)$.

8. If $\cos \theta = 0.6$ and θ in Q I, find $\sin \theta$.

9. Simplify $1 + \tan^2\theta$.

10. Simplify $(1 - \cos^2\theta) / \sin \theta$.

11. Verify $\cos(-\theta) \cdot \sec \theta = 1$.

12. Find the value of $\cos \theta$ if $\sin \theta = 5/13$ (Q II).

13. Simplify $\tan \theta \cdot \cos \theta$.

14. Why is $\sin^2\theta + \cos^2\theta = 1$ called Pythagorean?

15. Simplify $\sec^2\theta - \tan^2\theta$.

UNIT 4 · TRIGONOMETRY

Week 18 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. Simplify $1 - \sin^2\theta$.

- **(A)** $\cos^2\theta$
- **(B)** 1
- **(C)** $\tan^2\theta$
- **(D)** $\sin^2\theta$

Your answer: ____________

2. Simplify $\sec^2\theta - \tan^2\theta$.

- **(A)** $\tan^2\theta$
- **(B)** $\sec^2\theta$
- **(C)** 0
- **(D)** 1

Your answer: ____________

3. Simplify $\tan\theta \cdot \cot\theta$.

- **(A)** $\sin\theta$
- **(B)** $\cos\theta$
- **(C)** 1
- **(D)** 0

Your answer: ____________

4. $\sin(-\theta) = ?$

- **(A)** $-\cos\theta$
- **(B)** $-\sin\theta$
- **(C)** $\sin\theta$
- **(D)** $\cos\theta$

Your answer: ____________

5. $\cos(-\theta) = ?$

- **(A)** $-\sin\theta$
- **(B)** $\sin\theta$
- **(C)** $\cos\theta$
- **(D)** $-\cos\theta$

Your answer: ____________

6. Express $\tan\theta$ in terms of sin and cos.

- **(A)** $\sin\theta \cdot \cos\theta$
- **(B)** $\cos\theta / \sin\theta$
- **(C)** $\sin\theta / \cos\theta$

(D) 1/(sin θ · cos θ)

Your answer: ____________

7. If sin θ = 3/5 and θ is in Q I, find cos θ.

(A) 4/5
(B) 5/3
(C) 3/4
(D) −4/5

Your answer: ____________

8. Simplify $\sin^2\theta + \cos^2\theta$.

(A) sin θ + cos θ
(B) 1
(C) 0
(D) 2

Your answer: ____________

DID YOU KNOW?

In 1748, Euler discovered one of the most beautiful equations in mathematics: e^(iπ) + 1 = 0. The proof requires the trigonometric identities. Five fundamental constants — 0, 1, π, e, and i — combine in a single line. Physicist Richard Feynman called it "the most remarkable formula in mathematics." The same identities you are learning this week are the bridge.

UNIT 4 · TRIGONOMETRY

WEEK 19

Solving Trigonometric Equations

Trig equations have many solutions because trig functions are periodic. Use the unit circle to find solutions in one cycle, then add multiples of the period to find them all.

Learning Objectives

By the end of this week, you will be able to:

1. Define a trigonometric equation.
2. Solve basic sine, cosine, and tangent equations.
3. Use unit circle values to find solutions.
4. Apply algebraic methods to trig equations.
5. Find all solutions in a given interval.
6. Use identities to simplify equations.
7. Interpret solutions graphically.
8. Verify correctness of solutions.

Key Concepts

1. Trig equations have infinite solutions

Because sine, cosine, and tangent repeat forever, any trig equation that has one solution has infinitely many. We usually solve in the interval zero to 2 pi, then add the period.

2. Solving with the unit circle

For special values like one half, root two over two, and root three over two, read the angles straight off the unit circle.

3. Using identities to simplify first

Sometimes an equation needs algebra before you can read off solutions. Use Pythagorean and quotient identities to rewrite it in terms of a single function.

4. General solutions

To describe every solution to a trig equation, add multiples of the period to the solutions in one cycle.

5. Worked examples

Five worked trig equation examples.

Worked Examples

Example 1. Solve $\sin x = \sqrt{3}/2$ on $[0, 2\pi)$.

Solution: $x = \pi/3, 2\pi/3$.

Example 2. Solve $\cos x = 1/2$ on $[0, 2\pi)$.

Solution: $x = \pi/3, 5\pi/3$.

Example 3. Solve $\tan x = -1$ on $[0, 2\pi)$.

Solution: $x = 3\pi/4, 7\pi/4$.

UNIT 4 · TRIGONOMETRY

Week 19 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Solve $\sin x = \sqrt{3}/2$ on $[0, 2\pi)$.

2. Solve $\cos x = 1/2$ on $[0, 2\pi)$.

3. Solve $\tan x = -1$ on $[0, 2\pi)$.

4. Solve $\sin x = -\sqrt{2}/2$ on $[0, 2\pi)$.

5. Solve $\cos x = -1$ on $[0, 2\pi)$.

6. Solve $2 \sin^2 x = 1$ on $[0, 2\pi)$.

7. Solve $\sin x \cos x = 0$ on $[0, 2\pi)$.

8. Solve $\sin^2 x - \sin x = 0$ on $[0, 2\pi)$.

9. Solve $\sin x = 0.7$ (use calculator).

10. General solution of $\sin x = 0$.

11. Number of solutions of $\cos x = 1/2$ in $[0, 4\pi)$?

12. Solve $1 - 2 \sin x = 0$.

13. Solve $2 \cos x + 1 = 0$.

14. Solve $\tan^2 x = 3$.

15. Why does $\sin x = 2$ have no real solution?

UNIT 4 · TRIGONOMETRY

Week 19 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. Solve $\sin x = 1/2$ on $[0, 2\pi)$.

- **(A)** $x = \pi/6, 5\pi/6$
- **(B)** $x = \pi/3, 2\pi/3$
- **(C)** $x = \pi/2$
- **(D)** $x = \pi/4$

Your answer: ____________

2. Solve $\cos x = 0$ on $[0, 2\pi)$.

- **(A)** $x = \pi$
- **(B)** $x = \pi/2, 3\pi/2$
- **(C)** $x = 0, \pi$
- **(D)** $x = 2\pi$ only

Your answer: ____________

3. Solve $\tan x = 1$ on $[0, 2\pi)$.

- **(A)** $x = \pi/3, 5\pi/3$
- **(B)** $x = \pi/4$ only
- **(C)** $x = 0, \pi$
- **(D)** $x = \pi/4, 5\pi/4$

Your answer: ____________

4. How many solutions does $\sin x = 2$ have?

- **(A)** 2
- **(B)** infinite
- **(C)** 0
- **(D)** 1

Your answer: ____________

5. Solve $2 \sin x = 1$ on $[0, 2\pi)$.

- **(A)** $x = 1/2$
- **(B)** $x = \pi/6, 5\pi/6$
- **(C)** $x = 30°, 150°$
- **(D)** $x = \pi/3, 2\pi/3$

Your answer: ____________

6. General solution of $\cos x = 1$?

- **(A)** $x = 2\pi n$
- **(B)** $x = \pi + 2\pi n$
- **(C)** $x = \pi/2 + \pi n$

(D) $x = \pi n$

Your answer: ____________

7. Solve $1 - \cos^2 x = 0$ on $[0, 2\pi)$.

(A) $x = \pi/2, 3\pi/2$

(B) $x = 0$ only

(C) $x = 0, \pi$

(D) $x = \pi/4, 3\pi/4$

Your answer: ____________

8. Solve $\sin x = -1$ on $[0, 2\pi)$.

(A) $x = \pi$

(B) $x = 2\pi$

(C) $x = 3\pi/2$

(D) $x = \pi/2$

Your answer: ____________

DID YOU KNOW?

When your phone uses GPS to locate you, it solves a system of trig equations every second. Each satellite sends a timing signal; the differences in arrival time form trig equations whose solutions are your location. The same equations you are solving this week are running, billions of times per day, in every phone on Earth.

UNIT 4 · TRIGONOMETRY

WEEK 20

Applications & Unit Review

The Unit 4 capstone. Synthesise every trig tool — ratios, identities, equations, graphs, the unit circle, and the laws of sines and cosines — and apply them to real-world periodic phenomena.

Learning Objectives

By the end of this week, you will be able to:

1. Demonstrate understanding of core trigonometric concepts.
2. Apply trigonometry to real-world applications.
3. Solve right and oblique triangle problems.
4. Analyse and interpret trig graphs.
5. Solve trigonometric equations accurately.
6. Select appropriate solving strategies.
7. Identify and correct common errors.
8. Prepare for cumulative unit assessment.

Key Concepts

1. The trig toolkit

Eight weeks have given you a complete toolkit: triangle ratios, the unit circle, identities, equations, graphs, and the two laws for non-right triangles. Each tool fits a specific problem type.

2. Where trig appears in the world

Physics for waves and motion, engineering for forces and structures, navigation for angles and distances, architecture for heights and slopes, and data modelling for periodic trends.

3. Modelling periodic behaviour

To build a sinusoidal model: find the midline as average of max and min, amplitude as half the swing, and period as one full cycle. Plug into y equals a sine of b x minus c plus d.

4. Common errors to avoid

Make sure your calculator is in the right mode. Watch quadrant signs. Remember sin inverse is not 1 over sine. Always check for extraneous solutions.

5. Worked examples

Five cross-tool review examples.

Worked Examples

Example 1. Right triangle: $\theta = 60°$, hypotenuse = 8. Find both legs.

Solution: Opposite = $8 \cdot \sin 60° \approx 6.93$. Adjacent = $8 \cdot \cos 60° = 4$.

Example 2. In triangle ABC: $a = 7$, $b = 9$, $C = 40°$. Find c.

Solution: $c^2 = 49 + 81 - 126 \cdot \cos 40° \approx 33.5$. $c \approx 5.79$.

Example 3. Find $\sin(7\pi/6)$.

Solution: Reference $\pi/6$, Q III: $-1/2$.

UNIT 4 · TRIGONOMETRY

Week 20 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Right triangle: $\theta = 60°$, hypotenuse = 8. Find both legs.

2. In triangle ABC: $a = 7$, $b = 9$, $C = 40°$. Find c.

3. Find $\sin(7\pi/6)$.

4. Find $\cos(2\pi/3)$.

5. Solve $\sin x = 1$ on $[0, 2\pi)$.

6. Solve $2 \sin x - 1 = 0$ on $[0, 2\pi)$.

7. Solve $\tan x = \sqrt{3}$ on $[0, 2\pi)$.

8. Simplify $1 + \tan^2 x$.

9. For $y = 4 \sin(2x) - 3$, give amp, period, midline.

10. A bridge cable at 40° to horizontal, 25 m horizontal. Height?

11. Find area of triangle with $a = 5$, $b = 7$, $C = 60°$.

12. A pendulum height $h(t) = 0.5 \cos(2t) + 1$. Find max and min.

13. A tide max 6 m, min 1 m, period 12 hours. Write a model.

14. Solve $\cos^2 x = 1/4$ on $[0, 2\pi)$.

15. A plane flies at bearing 030° for 100 km, then 120° for 80 km. Find distance from start using Law of Cosines.

UNIT 4 · TRIGONOMETRY

Week 20 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. $\theta = 45°$, hypotenuse = 10. Find the opposite side.

- **(A)** 10
- **(B)** ≈ 7.07
- **(C)** 5
- **(D)** 14.14

Your answer: ____________

2. $\sin(\pi/3) = ?$

- **(A)** $\sqrt{2}/2$
- **(B)** $1/2$
- **(C)** $\sqrt{3}/2$
- **(D)** 1

Your answer: ____________

3. $\cos(\pi/4) = ?$

- **(A)** $1/2$
- **(B)** $\sqrt{2}/2$
- **(C)** $\sqrt{3}/2$
- **(D)** 0

Your answer: ____________

4. Solve $\cos x = 0$ on $[0, 2\pi)$.

- **(A)** $x = 2\pi$
- **(B)** $x = \pi$
- **(C)** $x = \pi/2, 3\pi/2$
- **(D)** $x = 0, \pi$

Your answer: ____________

5. Solve $\sin x = -1/2$ on $[0, 2\pi)$.

- **(A)** $x = \pi/3$
- **(B)** $x = 0, \pi$
- **(C)** $x = 7\pi/6, 11\pi/6$
- **(D)** $x = \pi/6, 5\pi/6$

Your answer: ____________

6. Simplify $1 - \cos^2 x$.

- **(A)** 1
- **(B)** $\sin^2 x$
- **(C)** $\cos^2 x$

(D) $\tan^2 x$

Your answer: ____________

7. A bridge cable forms 40° with horizontal. If horizontal distance is 25 m, find height.

(A) ≈ 19.15 m
(B) ≈ 20.98 m
(C) 40 m
(D) 25 m

Your answer: ____________

8. Which law solves a triangle with SAS data?

(A) Law of Cosines
(B) Law of Sines
(C) Pythagorean
(D) SOH-CAH-TOA

Your answer: ____________

DID YOU KNOW?

When you stand at the equator and watch the Sun rise and set, what you see is one full cycle of a sinusoid every day. When you fly to the Arctic Circle in summer, the Sun never sets — that is what happens to the sine model when the amplitude grows so large that the cycle never crosses zero. Every astronomy app on your phone uses the trig you have learned this unit to predict sunrise, sunset, and twilight for any location on Earth.

ANSWER KEY

Unit 4 · Trigonometry

This answer key covers every practice exercise and quiz question from Unit 4. For full step-by-step solutions to randomised practice generators (separate from the worksheet exercises printed here), refer to the BemandaSTEM Precalculus app.

ANSWER KEY

Week 13 — Trigonometric Ratios

Practice Exercises

1. **Q.** Find sin(θ) when opposite = 8 and hypotenuse = 10.
   A. 8/10 = 4/5 = 0.8.
2. **Q.** Find cos(θ) when adjacent = 5 and hypotenuse = 13.
   A. 5/13 ≈ 0.385.
3. **Q.** Find tan(θ) when opposite = 9 and adjacent = 12.
   A. 9/12 = 3/4 = 0.75.
4. **Q.** If sin(θ) = 0.6 and hypotenuse = 20, find the opposite.
   A. Opposite = 0.6 · 20 = 12.
5. **Q.** If cos(θ) = 0.8 and hypotenuse = 25, find the adjacent.
   A. Adjacent = 0.8 · 25 = 20.
6. **Q.** Find θ if tan(θ) = 2.
   A. θ = $\tan^{-1}(2)$ ≈ 63.43°.
7. **Q.** Find θ if sin(θ) = 0.5.
   A. θ = 30°.
8. **Q.** A ladder is leaning against a wall at 60° angle with the ground. The ladder is 10 m long. Find the height reached.
   A. Height = 10 sin(60°) ≈ 8.66 m.
9. **Q.** A ramp rises 2 m over a horizontal distance of 8 m. Find the angle of elevation.
   A. tan(θ) = 2/8 = 0.25. θ ≈ 14.04°.
10. **Q.** A tower is observed from 50 m away. Angle of elevation to the top is 30°. Find the tower\\u2019s height.
    A. Height = 50 · tan(30°) ≈ 28.87 m.
11. **Q.** From the top of a 40 m cliff, a boat is observed at 25° angle of depression. Find the horizontal distance.
    A. tan(25°) = 40 / distance → distance = 40 / tan(25°) ≈ 85.78 m.
12. **Q.** Find sin(θ) where opposite = 3 and adjacent = 4.
    A. Hypotenuse = √(9 + 16) = 5. sin(θ) = 3/5 = 0.6.
13. **Q.** A kite is on a 50 m string at 70° angle from the ground. Height?
    A. Height = 50 sin(70°) ≈ 46.98 m.
14. **Q.** For sin(θ) = 0.5, cos(θ) = 0.5: is this possible?
    A. No — $\sin^2 + \cos^2$ = 0.25 + 0.25 = 0.5 ≠ 1.
15. **Q.** A 20 m flagpole casts a 12 m shadow. Find the sun\\u2019s angle of elevation.
    A. tan(θ) = 20/12 ≈ 1.667. θ ≈ 59.04°.

Quiz Answers

1. **Answer: (D) 5/13**
   Reason: SOH: sin(θ) = opposite/hypotenuse = 5/13.

2. **Answer: (D) 8/17**

   Reason: CAH: $\cos(\theta)$ = adjacent/hypotenuse = 8/17.

3. **Answer: (D) 7/24**

   Reason: TOA: $\tan(\theta)$ = opposite/adjacent = 7/24.

4. **Answer: (C) 10**

   Reason: Opposite = $0.5 \cdot 20 = 10$.

5. **Answer: (C) 45°**

   Reason: $\tan(45°) = 1$, so $\theta = 45°$.

6. **Answer: (A) 8.66 m**

   Reason: Height = $10 \cdot \sin(60°) = 10 \cdot 0.866 \approx 8.66$ m.

7. **Answer: (C) The side across from θ**

   Reason: "Opposite" means across from your reference angle.

8. **Answer: (D) 30°**

   Reason: $\sin(30°) = 0.5$, so $\sin^{-1}(0.5) = 30°$.

ANSWER KEY

Week 14 — Right Triangle Problems

Practice Exercises

1. **Q.** Find the hypotenuse of a right triangle with legs 9 and 12.
   A. c = √(81 + 144) = √225 = 15.
2. **Q.** Find the missing leg if hypotenuse = 17 and one leg = 8.
   A. b = √(289 − 64) = √225 = 15. (8-15-17 triple.)
3. **Q.** Find the missing leg if hypotenuse = 25 and one leg = 7.
   A. b = √(625 − 49) = √576 = 24. (7-24-25 triple.)
4. **Q.** θ = 45°, hypotenuse = 14. Find both legs.
   A. Each leg = 14 · √2/2 = 7√2 ≈ 9.90.
5. **Q.** θ = 30°, hypotenuse = 20. Find the opposite leg.
   A. Opposite = 20 · 0.5 = 10.
6. **Q.** θ = 60°, hypotenuse = 12. Find the adjacent leg.
   A. Adjacent = 12 · cos(60°) = 12 · 0.5 = 6.
7. **Q.** A 50 m tower is observed from 30° elevation. Find the distance from base.
   A. tan(30°) = 50/d → d = 50/tan(30°) ≈ 86.60 m.
8. **Q.** A 5 m ramp at 20° angle. Find the height.
   A. Height = 5 · sin(20°) ≈ 1.71 m.
9. **Q.** From the top of a 30 m cliff, a boat is seen at 25° depression. Distance to boat?
   A. tan(25°) = 30/d → d = 30/tan(25°) ≈ 64.34 m.
10. **Q.** sin(θ) = 5/13. Find tan(θ) using Pythagorean.
    A. If opposite=5, hypotenuse=13, then adjacent = √(169−25) = 12. tan(θ) = 5/12.
11. **Q.** A pole 8 m tall casts a 6 m shadow. Find the sun\\u2019s elevation.
    A. tan(θ) = 8/6, so θ ≈ 53.13°.
12. **Q.** A 10 m ladder reaches 8 m up a wall. Distance from wall?
    A. d = √(100 − 64) = √36 = 6 m.
13. **Q.** Find the angle of a 5-12-13 right triangle opposite the side of length 5.
    A. sin(θ) = 5/13, θ ≈ 22.62°.
14. **Q.** A right triangle has angles 30°, 60°, 90°. If the shortest side is 4, find the other sides.
    A. In a 30-60-90 triangle: sides in ratio 1:√3:2. Sides 4, 4√3 ≈ 6.93, 8.
15. **Q.** A flagpole leans 5° from vertical and casts a shadow at noon. Is this a right triangle problem? Why or why not?
    A. No — without a 90° angle, neither trig ratios nor Pythagorean directly apply. Could be modelled with Law of Sines/Cosines instead.

Quiz Answers

1. **Answer: (B)** **10**

Reason: $c^2 = 36 + 64 = 100$, so $c = 10$.

2. **Answer: (D) 12**

   Reason: $b^2 = 169 - 25 = 144$, so $b = 12$. (5-12-13 triple.)

3. **Answer: (A) $10 \cdot \sqrt{2}/2 \approx 7.07$**

   Reason: Opposite = $10 \cdot \sin(45°) = 10 \cdot \sqrt{2}/2 \approx 7.07$.

4. **Answer: (C) ≈ 12.99 m**

   Reason: Height = $15 \cdot \sin(60°) = 15 \cdot 0.866 \approx 12.99$ m.

5. **Answer: (A) ≈ 36.87°**

   Reason: $\theta = \tan^{-1}(0.75) \approx 36.87°$.

6. **Answer: (A) 20 m**

   Reason: $\tan(45°) = h/20$, so $h = 20 \cdot 1 = 20$ m.

7. **Answer: (D) ≈ 36.87°**

   Reason: $\tan(\theta) = 3/4$. $\theta \approx 36.87°$.

8. **Answer: (A) Pythagorean theorem**

   Reason: No angle involved → Pythagorean theorem.

ANSWER KEY

Week 15 — Oblique Triangles

Practice Exercises

1. **Q.** $A = 40°$, $B = 70°$, $a = 15$. Find b.

   A. $b = 15 \sin(70°)/\sin(40°) \approx 21.93$.

2. **Q.** $A = 50°$, $b = 12$, $c = 18$. Find a (Law of Cosines).

   A. $a^2 = 144 + 324 - 432 \cos(50°) \approx 190.36$. $a \approx 13.80$.

3. **Q.** $a = 7$, $b = 9$, $C = 60°$. Find c.

   A. $c^2 = 49 + 81 - 126 \cdot 0.5 = 67$. $c \approx 8.19$.

4. **Q.** $a = 9$, $b = 10$, $c = 12$. Find angle A.

   A. $\cos(A) = (100 + 144 - 81)/(240) = 163/240$. $A \approx 47.16°$.

5. **Q.** Find area with $a = 7$, $b = 9$, $C = 45°$.

   A. Area $= (1/2)(7)(9)\sin(45°) \approx 22.27$.

6. **Q.** Find area with $a = 10$, $b = 8$, $C = 60°$.

   A. Area $= (1/2)(10)(8)\sin(60°) \approx 34.64$.

7. **Q.** A surveyor measures two sides of a triangular plot: 50 m and 80 m with included angle 70°. Find the third side.

   A. $c^2 = 2500 + 6400 - 8000 \cos(70°) \approx 6164.6$. $c \approx 78.51$ m.

8. **Q.** Two ships travel from port; one 25 km north, the other 40 km at 60° east of north. Distance between?

   A. $d^2 = 25^2 + 40^2 - 2 \cdot 25 \cdot 40 \cdot \cos(60°) = 625 + 1600 - 1000 = 1225$. $d = 35$ km.

9. **Q.** A triangle has $A = 35°$, $a = 12$, $b = 18$. Find B (Law of Sines).

   A. $\sin(B)/18 = \sin(35°)/12$. $\sin(B) = 18 \cdot \sin(35°)/12 \approx 0.860$. $B \approx 59.36°$ (or 120.64° — ambiguous).

10. **Q.** In a triangle, $A = 60°$, $B = 80°$, $c = 14$. Find a.

    A. $C = 180° - 60° - 80° = 40°$. $a = 14 \sin(60°)/\sin(40°) \approx 18.86$.

11. **Q.** Find C if $\cos(C) = 0.5$.

    A. $C = 60°$.

12. **Q.** Two sides 12 and 18, area = 50 square units. Find the included angle.

    A. $50 = (1/2)(12)(18)\sin(C) \rightarrow \sin(C) = 100/216 \approx 0.463$. $C \approx 27.55°$.

13. **Q.** If $A = 30°$, $B = 60°$, what type of triangle is it (angle-wise)?

    A. Right triangle (since $30 + 60 + 90 = 180$).

14. **Q.** For SSA case with $A = 30°$, $a = 5$, $b = 12$, why is it problematic?

    A. $\sin(B) = 12 \cdot \sin(30°)/5 = 1.2 > 1$ — impossible. No triangle exists.

15. **Q.** Find the area of a triangle with sides 6, 8, and included angle 90°.

    A. Area $= (1/2)(6)(8)\sin(90°) = 24$ — same as (1/2) base × height.

Quiz Answers

1. **Answer: (C) SAS (two sides and included angle)**

   Reason: SAS and SSS require Law of Cosines. ASA and AAS use Law of Sines.

2. **Answer: (D) ≈ 17.32**

   Reason: $\sin(30°)/10 = \sin(60°)/b \rightarrow b = 10\sin(60°)/\sin(30°) = 10 \cdot \sqrt{3} \approx 17.32$.

3. **Answer: (B) ≈ 6.24**

   Reason: $c^2 = 25 + 49 - 2\cdot5\cdot7\cdot0.5 = 39$. $c \approx 6.24$.

4. **Answer: (D) 30**

   Reason: $\text{Area} = (1/2)(10)(12)\sin(30°) = 60 \cdot 0.5 = 30$.

5. **Answer: (C) 60°**

   Reason: Sum is 180°: $C = 180° - 50° - 70° = 60°$.

6. **Answer: (D) 50 km**

   Reason: 90° = right angle, so use Pythagorean: $\sqrt{(30^2 + 40^2)} = 50$ km. (Or Law of Cosines reduces to this.)

7. **Answer: (C) Law of Cosines**

   Reason: SSS → Law of Cosines: $\cos(A) = (b^2 + c^2 - a^2)/(2bc)$.

8. **Answer: (B) They can have 0, 1, or 2 triangles (ambiguous)**

   Reason: SSA is the ambiguous case — given data may match no triangle, one, or two triangles.

ANSWER KEY

Week 16 — Unit Circle Basics

Practice Exercises

1. **Q.** Find coordinates of 30°.
   A. $(\sqrt{3}/2, 1/2)$.
2. **Q.** Find coordinates of 45°.
   A. $(\sqrt{2}/2, \sqrt{2}/2)$.
3. **Q.** Find coordinates of 90°.
   A. (0, 1).
4. **Q.** Convert 180° to radians.
   A. π radians.
5. **Q.** Convert 60° to radians.
   A. π/3 radians.
6. **Q.** Convert π/2 to degrees.
   A. 90°.
7. **Q.** Convert 5π/6 to degrees.
   A. $5\pi/6 \cdot 180/\pi = 150°$.
8. **Q.** In which quadrant is 120°? What signs do sin, cos, tan have?
   A. QII. sin: +, cos: −, tan: −.
9. **Q.** In which quadrant is 225°? Signs?
   A. QIII. sin: −, cos: −, tan: +.
10. **Q.** Find the reference angle for 135°.
    A. $180° - 135° = 45°$.
11. **Q.** Find the reference angle for 300°.
    A. $360° - 300° = 60°$.
12. **Q.** Find sin(150°).
    A. Reference angle 30°, QII: $\sin > 0$. $\sin(150°) = 1/2$.
13. **Q.** Find cos(240°).
    A. Reference angle 60°, QIII: $\cos < 0$. $\cos(240°) = -1/2$.
14. **Q.** Find tan(315°).
    A. Reference angle 45°, QIV: $\tan < 0$. $\tan(315°) = -1$.
15. **Q.** Verify $\sin^2(30°) + \cos^2(30°) = 1$.
    A. $(1/2)^2 + (\sqrt{3}/2)^2 = 1/4 + 3/4 = 1$. ✓

Quiz Answers

1. **Answer: (A) $(1/2, \sqrt{3}/2)$**
   Reason: At 60°: $(\cos 60°, \sin 60°) = (1/2, \sqrt{3}/2)$.
2. **Answer: (D) π/2**
   Reason: $90° \cdot \pi/180 = \pi/2$.

3. **Answer: (C) 60°**

   Reason: $\pi/3 \cdot 180/\pi = 60°$.

4. **Answer: (A) III**

   Reason: $180° < 210° < 270°$, so Quadrant III.

5. **Answer: (A) negative**

   Reason: 225° is in Quadrant III, where only tangent is positive. Cosine is negative.

6. **Answer: (B) 30°**

   Reason: $180° - 150° = 30°$.

7. **Answer: (B) 0**

   Reason: At 180°: point is $(-1, 0)$. sin = y-coordinate = 0.

8. **Answer: (C) 0**

   Reason: At 270°: point is $(0, -1)$. cos = x-coordinate = 0.

ANSWER KEY

Week 17 — Graphing Trigonometric Functions

Practice Exercises

1. **Q.** Find amplitude of y = 2 sin(x).

   A. Amplitude = 2.

2. **Q.** Find period of y = sin(4x).

   A. 2π/4 = π/2.

3. **Q.** Find midline of y = sin(x) + 5.

   A. y = 5.

4. **Q.** For y = 3 cos(2x) − 1, give amp, period, midline.

   A. Amp 3, period π, midline y = −1.

5. **Q.** Find max of y = 2 sin(x) + 3.

   A. 3 + 2 = 5.

6. **Q.** Find min of y = 2 sin(x) + 3.

   A. 3 − 2 = 1.

7. **Q.** sin(0) = ?

   A. 0.

8. **Q.** cos(0) = ?

   A. 1.

9. **Q.** sin(π) = ?

   A. 0.

10. **Q.** A tide goes from 0.5 m to 4.5 m over 12 hours. Model the height.

    A. Amp = 2, midline = 2.5, period 12 → b = 2π/12 = π/6. h(t) = 2 sin((π/6)t) + 2.5.

11. **Q.** Compare y = sin(x) and y = cos(x). How are they related?

    A. cos(x) = sin(x + π/2). Cosine is sine shifted left by π/2.

12. **Q.** Find period of y = tan(2x).

    A. π/2.

13. **Q.** Is y = sin(x) an even or odd function?

    A. Odd: sin(−x) = −sin(x).

14. **Q.** Is y = cos(x) an even or odd function?

    A. Even: cos(−x) = cos(x).

15. **Q.** A piston moves up and down 30 cm peak-to-peak, 60 cycles per second. Amplitude and period?

    A. Amplitude = 15 cm. Period = 1/60 second.

Quiz Answers

1. **Answer: (D) 4**

   Reason: Amplitude = $|a| = 4$*.*

2. **Answer: (D) 2π/3**

Reason: Period = 2π/b = 2π/3.

3. **Answer: (C) $y = -2$**

   Reason: Vertical shift −2 → midline y = −2.

4. **Answer: (B) 6**

   Reason: Max = midline + amp = 1 + 5 = 6.

5. **Answer: (B) 1**

   Reason: At π/2, sine reaches its maximum of 1.

6. **Answer: (C) −1**

   Reason: At π, cosine reaches its minimum of −1.

7. **Answer: (D) π**

   Reason: Tangent has period π, not 2π like sine/cosine.

8. **Answer: (D) 10 m**

   Reason: Amp = (max − min)/2 = (22 − 2)/2 = 10 m.

ANSWER KEY

Week 18 — Trigonometric Identities

Practice Exercises

1. **Q.** Simplify $1 - \cos^2\theta$.

   A. $\sin^2\theta$.
2. **Q.** Simplify $\sin\theta / \cos\theta$.

   A. $\tan\theta$.
3. **Q.** Express $\sec\theta$ in basic form.

   A. $1/\cos\theta$.
4. **Q.** Express $\cot\theta$ in basic form.

   A. $\cos\theta / \sin\theta$ or $1/\tan\theta$.
5. **Q.** Verify $\sin\theta / \cos\theta = \tan\theta$.

   A. Direct from quotient identity.
6. **Q.** Find $\sin(-45°)$.

   A. $-\sin(45°) = -\sqrt{2}/2$.
7. **Q.** Find $\tan(-60°)$.

   A. $-\tan(60°) = -\sqrt{3}$.
8. **Q.** If $\cos\theta = 0.6$ and θ in Q I, find $\sin\theta$.

   A. $\sin\theta = \sqrt{(1 - 0.36)} = \sqrt{0.64} = 0.8$.
9. **Q.** Simplify $1 + \tan^2\theta$.

   A. $\sec^2\theta$.
10. **Q.** Simplify $(1 - \cos^2\theta) / \sin\theta$.

    A. $\sin^2\theta / \sin\theta = \sin\theta$.
11. **Q.** Verify $\cos(-\theta) \cdot \sec\theta = 1$.

    A. $\cos\theta \cdot (1/\cos\theta) = 1$.
12. **Q.** Find the value of $\cos\theta$ if $\sin\theta = 5/13$ (Q II).

    A. $\cos^2\theta = 1 - 25/169 = 144/169$. $\cos\theta = \pm 12/13$. In Q II, $\cos < 0$: $\cos\theta = -12/13$.
13. **Q.** Simplify $\tan\theta \cdot \cos\theta$.

    A. $(\sin\theta/\cos\theta) \cdot \cos\theta = \sin\theta$.
14. **Q.** Why is $\sin^2\theta + \cos^2\theta = 1$ called Pythagorean?

    A. On the unit circle, the point $(\cos\theta, \sin\theta)$ satisfies $x^2 + y^2 = 1$ — the Pythagorean theorem.
15. **Q.** Simplify $\sec^2\theta - \tan^2\theta$.

    A. 1 (from $1 + \tan^2\theta = \sec^2\theta$).

Quiz Answers

1. **Answer: (A) $\cos^2\theta$**

   Reason: Pythagorean: $\sin^2\theta + \cos^2\theta = 1 \rightarrow 1 - \sin^2\theta = \cos^2\theta$.
2. **Answer: (D) 1**

Reason: Pythagorean: $1 + \tan^2\theta = \sec^2\theta \rightarrow \sec^2\theta - \tan^2\theta = 1$.

3. **Answer: (C) 1**

   Reason: $\cot\theta = 1/\tan\theta$, *so product* $= 1$.

4. **Answer: (B)** $-\sin\theta$

   Reason: Sine is an odd function.

5. **Answer: (C)** $\cos\theta$

   Reason: Cosine is an even function.

6. **Answer: (C)** $\sin\theta / \cos\theta$

   Reason: Quotient identity.

7. **Answer: (A) 4/5**

   Reason: $\cos^2\theta = 1 - 9/25 = 16/25$. $\cos\theta = 4/5$ *(positive in Q I).*

8. **Answer: (B) 1**

   Reason: Fundamental Pythagorean identity.

ANSWER KEY

Week 19 — Solving Trigonometric Equations

Practice Exercises

1. **Q.** Solve $\sin x = \sqrt{3}/2$ on $[0, 2\pi)$.
   A. $x = \pi/3, 2\pi/3$.
2. **Q.** Solve $\cos x = 1/2$ on $[0, 2\pi)$.
   A. $x = \pi/3, 5\pi/3$.
3. **Q.** Solve $\tan x = -1$ on $[0, 2\pi)$.
   A. $x = 3\pi/4, 7\pi/4$.
4. **Q.** Solve $\sin x = -\sqrt{2}/2$ on $[0, 2\pi)$.
   A. $x = 5\pi/4, 7\pi/4$.
5. **Q.** Solve $\cos x = -1$ on $[0, 2\pi)$.
   A. $x = \pi$.
6. **Q.** Solve $2 \sin^2 x = 1$ on $[0, 2\pi)$.
   A. $\sin x = \pm\sqrt{2}/2 \rightarrow x = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.
7. **Q.** Solve $\sin x \cos x = 0$ on $[0, 2\pi)$.
   A. $\sin x = 0$ ($x = 0, \pi$) or $\cos x = 0$ ($x = \pi/2, 3\pi/2$).
8. **Q.** Solve $\sin^2 x - \sin x = 0$ on $[0, 2\pi)$.
   A. $\sin x(\sin x - 1) = 0 \rightarrow x = 0, \pi, \pi/2$.
9. **Q.** Solve $\sin x = 0.7$ (use calculator).
   A. $x = \sin^{-1}(0.7) \approx 0.7754$ rad, and $\pi - 0.7754 \approx 2.366$ rad.
10. **Q.** General solution of $\sin x = 0$.
    A. $x = \pi n$, n any integer.
11. **Q.** Number of solutions of $\cos x = 1/2$ in $[0, 4\pi)$?
    A. 4 (two per period; two periods).
12. **Q.** Solve $1 - 2 \sin x = 0$.
    A. $\sin x = 1/2 \rightarrow x = \pi/6, 5\pi/6$.
13. **Q.** Solve $2 \cos x + 1 = 0$.
    A. $\cos x = -1/2 \rightarrow x = 2\pi/3, 4\pi/3$.
14. **Q.** Solve $\tan^2 x = 3$.
    A. $\tan x = \pm\sqrt{3} \rightarrow x = \pi/3, 2\pi/3, 4\pi/3, 5\pi/3$.
15. **Q.** Why does $\sin x = 2$ have no real solution?
    A. $\sin x$ is bounded in $[-1, 1]$.

Quiz Answers

1. **Answer: (A) $x = \pi/6, 5\pi/6$**
   Reason: Sine = 1/2 at π/6 and 5π/6.
2. **Answer: (B) $x = \pi/2, 3\pi/2$**
   Reason: cos = 0 at the top and bottom of the unit circle.

3. **Answer: (D) $x = \pi/4, 5\pi/4$**

   Reason: $\tan = 1$ at $\pi/4$; period π gives $\pi/4 + \pi = 5\pi/4$.

4. **Answer: (C) 0**

   Reason: $\sin x$ is bounded between -1 and 1, so 2 is impossible.

5. **Answer: (B) $x = \pi/6, 5\pi/6$**

   Reason: $\sin x = 1/2 \rightarrow x = \pi/6, 5\pi/6$.

6. **Answer: (A) $x = 2\pi n$**

   Reason: $\cos x = 1$ at $x = 0$; period $2\pi \rightarrow$ general solution $x = 2\pi n$.

7. **Answer: (C) $x = 0, \pi$**

   Reason: $\cos^2 x = 1 \rightarrow \cos x = \pm 1 \rightarrow x = 0$ or π.

8. **Answer: (C) $x = 3\pi/2$**

   Reason: sin reaches its minimum at $3\pi/2$.

ANSWER KEY

Week 20 — Applications & Unit Review

Practice Exercises

1. **Q.** Right triangle: $\theta = 60°$, hypotenuse = 8. Find both legs.
   A. Opposite = $8 \cdot \sin 60° \approx 6.93$. Adjacent = $8 \cdot \cos 60° = 4$.
2. **Q.** In triangle ABC: $a = 7$, $b = 9$, $C = 40°$. Find c.
   A. $c^2 = 49 + 81 - 126 \cdot \cos 40° \approx 33.5$. $c \approx 5.79$.
3. **Q.** Find $\sin(7\pi/6)$.
   A. Reference $\pi/6$, Q III: $-1/2$.
4. **Q.** Find $\cos(2\pi/3)$.
   A. Reference $\pi/3$, Q II: $-1/2$.
5. **Q.** Solve $\sin x = 1$ on $[0, 2\pi)$.
   A. $x = \pi/2$.
6. **Q.** Solve $2 \sin x - 1 = 0$ on $[0, 2\pi)$.
   A. $\sin x = 1/2 \rightarrow x = \pi/6, 5\pi/6$.
7. **Q.** Solve $\tan x = \sqrt{3}$ on $[0, 2\pi)$.
   A. $x = \pi/3, 4\pi/3$.
8. **Q.** Simplify $1 + \tan^2 x$.
   A. $\sec^2 x$.
9. **Q.** For $y = 4 \sin(2x) - 3$, give amp, period, midline.
   A. Amp 4, period π, midline $y = -3$.
10. **Q.** A bridge cable at 40° to horizontal, 25 m horizontal. Height?
    A. $h = 25 \tan 40° \approx 20.98$ m.
11. **Q.** Find area of triangle with $a = 5$, $b = 7$, $C = 60°$.
    A. $(1/2) \cdot 5 \cdot 7 \cdot \sin 60° \approx 15.16$.
12. **Q.** A pendulum height $h(t) = 0.5 \cos(2t) + 1$. Find max and min.
    A. Max = 1.5; min = 0.5.
13. **Q.** A tide max 6 m, min 1 m, period 12 hours. Write a model.
    A. Amp 2.5, midline 3.5, $b = \pi/6$. $h(t) = 2.5 \sin((\pi/6)t) + 3.5$.
14. **Q.** Solve $\cos^2 x = 1/4$ on $[0, 2\pi)$.
    A. $\cos x = \pm 1/2 \rightarrow x = \pi/3, 2\pi/3, 4\pi/3, 5\pi/3$.
15. **Q.** A plane flies at bearing 030° for 100 km, then 120° for 80 km. Find distance from start using Law of Cosines.
    A. Angle between paths = $180° - (120° - 30°) = 90°$. $d^2 = 100^2 + 80^2 = 16400$. $d \approx 128.06$ km.

Quiz Answers

1. **Answer: (B) ≈ 7.07**
   Reason: $10 \cdot \sin(45°) = 10 \cdot \sqrt{2}/2 \approx 7.07$.
2. **Answer: (C) $\sqrt{3}/2$**

Reason: $\sin(60°) = \sqrt{3}/2$.

3. **Answer: (B) $\sqrt{2}/2$**

   Reason: $\cos(45°) = \sqrt{2}/2$.

4. **Answer: (C) $x = \pi/2, 3\pi/2$**

   Reason: Cosine is zero at the top and bottom of the unit circle.

5. **Answer: (C) $x = 7\pi/6, 11\pi/6$**

   Reason: Sine is negative in Q III and Q IV: $7\pi/6$ and $11\pi/6$.

6. **Answer: (B) $\sin^2 x$**

   Reason: Pythagorean: $\sin^2 x + \cos^2 x = 1 \to 1 - \cos^2 x = \sin^2 x$.

7. **Answer: (B) ≈ 20.98 m**

   Reason: $h = 25 \cdot \tan(40°) \approx 20.98$ m.

8. **Answer: (A) Law of Cosines**

   Reason: SAS → Law of Cosines.

UNIT 5
Analytic Geometry

UNIT OVERVIEW

Algebra meets geometry on the coordinate plane. Master circles, the three conic sections (parabolas, ellipses, hyperbolas), and the workhorse formulas — distance, midpoint, slope — that connect points to shapes to real-world systems like satellite dishes, planetary orbits, and GPS navigation.

Weeks in this unit:

Week 21 — *Circles*
Week 22 — *Parabolas and Ellipses*
Week 23 — *Hyperbolas*
Week 24 — *Distance, Midpoint, and Slope Formulas*
Week 25 — *Conic Sections Applications*

UNIT 5 · ANALYTIC GEOMETRY

WEEK 21

Circles

The first conic section — and the simplest. Master the equation of a circle, the parts that define it (radius, diameter, chord, tangent), and the formulas for circumference and area.

Learning Objectives

By the end of this week, you will be able to:

1. Define a circle and its key components.
2. Calculate circumference and area of a circle.
3. Identify and use parts of a circle.
4. Apply basic circle equations in coordinate form.
5. Solve problems involving arcs and sectors.
6. Understand properties of tangents.
7. Relate angles to arcs in a circle.
8. Apply circle concepts to real-world problems.

Key Concepts

1. What is a circle?

A circle is the set of all points in a plane that are the same distance from a fixed point called the centre. That single fixed distance is the radius.

2. Circumference and area

Circumference is the distance around a circle, equal to 2 pi r. Area is the space inside, equal to pi r squared.

3. The equation of a circle

A circle of radius r centred at the point h k has the equation x minus h squared plus y minus k squared equals r squared.

4. Tangents and the perpendicular property

A tangent line touches a circle at exactly one point. At that point, the tangent is always perpendicular to the radius.

5. Worked examples

Five worked circle examples.

Worked Examples

Example 1. Find the circumference of a circle with $r = 7$.

Solution: $C = 14\pi \approx 43.98$.

Example 2. Find the circumference of a circle with $r = 12$.

Solution: $C = 24\pi \approx 75.40$.

Example 3. Find the area of a circle with $r = 6$.

Solution: $A = 36\pi \approx 113.10$.

UNIT 5 · ANALYTIC GEOMETRY

Week 21 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Find the circumference of a circle with $r = 7$.

2. Find the circumference of a circle with $r = 12$.

3. Find the area of a circle with $r = 6$.

4. Find the area of a circle with $r = 9$.

5. Find the diameter of a circle with $r = 15$.

6. Write equation of circle centred at origin, $r = 5$.

7. Write equation of circle centred at $(3, -2)$, $r = 4$.

8. Write equation of circle centred at (0, 4), $r = 7$.

9. Find centre and radius of $(x - 5)^2 + (y + 1)^2 = 36$.

10. Find centre and radius of $x^2 + y^2 = 81$.

11. A circular fountain has radius 4 m. Find circumference.

12. Find area of the same fountain.

13. A circular track has circumference 100 m. Find radius.

14. A circular field has area 100π m^2. Find radius.

15. A circle is tangent to the x-axis at (3, 0) and has radius 4. Find equation.

UNIT 5 · ANALYTIC GEOMETRY

Week 21 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. Circumference of a circle with radius 7 (in terms of π)?

(A) 7π
(B) 21π
(C) 49π
(D) 14π
Your answer: ____________

2. Area of a circle with radius 6 (in terms of π)?

(A) 36π
(B) 12π
(C) 6π
(D) 18π
Your answer: ____________

3. A circle has diameter 20. Its radius is…

(A) 10
(B) 40
(C) 20
(D) 5
Your answer: ____________

4. Equation of circle centred at origin with radius 5?

(A) $x^2 + y^2 = 10$
(B) $x^2 + y^2 = 5$
(C) $x + y = 5$
(D) $x^2 + y^2 = 25$
Your answer: ____________

5. Equation of circle centred at (3, −2) with radius 4?

(A) $(x - 3)^2 + (y + 2)^2 = 16$
(B) $(x - 3)^2 + (y - 2)^2 = 16$
(C) $(x - 3)^2 + (y + 2)^2 = 4$
(D) $(x + 3)^2 + (y - 2)^2 = 16$
Your answer: ____________

6. For $(x + 1)^2 + (y - 5)^2 = 49$, the centre is…

(A) (1, −5)
(B) (−1, 5)
(C) (−1, −5)

(D) (1, 5)
Your answer: ____________

7. For $(x - 2)^2 + (y + 3)^2 = 25$, the radius is…

(A) 12.5
(B) 25
(C) 5
(D) $\sqrt{5}$
Your answer: ____________

8. A tangent to a circle at point P meets the radius at P in what angle?

(A) 90°
(B) 180°
(C) 60°
(D) 45°
Your answer: ____________

DID YOU KNOW?

The first known calculation of π was made by the ancient Babylonians around 1900 BCE — they used 3.125, which is wrong by only 0.5%. Archimedes, around 250 BCE, used inscribed and circumscribed polygons with 96 sides to prove π lies between 223/71 and 22/7 — a brilliant proof still taught today. Today, π has been computed to over 100 trillion digits, but for almost all engineering applications, eight digits (3.14159265) are enough.

UNIT 5 · ANALYTIC GEOMETRY

WEEK 22

Parabolas and Ellipses

Two of the four conic sections: parabolas (one focus, satellite dishes) and ellipses (two foci, planetary orbits). Learn the standard forms, key features, and how to read a graph from an equation.

Learning Objectives

By the end of this week, you will be able to:

1. Define parabolas and ellipses as conic sections.
2. Identify key features of parabolas and ellipses.
3. Write equations of parabolas and ellipses in standard form.
4. Graph basic parabolas and ellipses.
5. Interpret real-world applications of conic sections.
6. Compare properties of parabolas and ellipses.
7. Solve basic problems involving conic equations.
8. Recognise transformations in conic graphs.

Key Concepts

1. What are conic sections?

Conic sections are the four curves you get when you slice a cone with a flat plane: circles, parabolas, ellipses, and hyperbolas. Each one represents a different distance relationship.

2. The parabola

A parabola is the set of all points equidistant from a fixed point (the focus) and a fixed line (the directrix). It has a vertex, an axis of symmetry, and one focus.

3. The ellipse

An ellipse is the set of all points where the sum of distances to two foci is constant. It looks like a stretched circle. It has a centre, two vertices on the major axis, two co-vertices on the minor axis, and two foci.

4. Reading conic equations

Look at the signs and structure of the equation to identify which conic it is. Pluses give circles or ellipses; one squared term gives a parabola; minuses give hyperbolas.

5. Worked examples

Five worked examples covering parabolas and ellipses.

Worked Examples

Example 1. Find the vertex of $(x - 3)^2 = 12(y + 2)$.

Solution: Vertex $(3, -2)$. Opens upward.

Example 2. Find the vertex of $(y + 1)^2 = -8(x - 4)$.

Solution: Vertex $(4, -1)$. Opens to the left.

Example 3. Rewrite $y = x^2 - 6x + 5$ in vertex form and find vertex.

Solution: $y = (x - 3)^2 - 4$. Vertex $(3, -4)$.

UNIT 5 · ANALYTIC GEOMETRY

Week 22 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Find the vertex of $(x - 3)^2 = 12(y + 2)$.

2. Find the vertex of $(y + 1)^2 = -8(x - 4)$.

3. Rewrite $y = x^2 - 6x + 5$ in vertex form and find vertex.

4. Rewrite $y = -x^2 + 4x + 1$ in vertex form.

5. Centre of $x^2/49 + y^2/16 = 1$?

6. Vertices of $x^2/25 + y^2/9 = 1$?

7. Co-vertices of the same ellipse?

8. Identify $(x - 2)^2/9 + (y + 3)^2/4 = 1$.

9. Identify $x^2 + y^2 = 25$.

10. Identify $(y - 1)^2 = 8(x + 2)$.

11. Why are parabolas used in headlights?

12. Why are ellipses used to describe planetary orbits?

13. In a whispering gallery, a person at one focus can clearly hear someone at the other focus. Why?

14. A parabola has vertex (0, 0) and focus (0, 2). Find its equation.

15. An ellipse has centre (0,0), vertices at (±10, 0), and co-vertices at (0, ±6). Find its equation.

UNIT 5 · ANALYTIC GEOMETRY

Week 22 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. A parabola is defined as the set of points equidistant from…

(A) A centre
(B) A vertex
(C) A focus and a directrix
(D) Two foci
Your answer: ____________

2. For $(x - 4)^2 = 12(y + 2)$, the vertex is…

(A) (4, −2)
(B) (4, 2)
(C) (−4, 2)
(D) (0, 0)
Your answer: ____________

3. For $y^2 = -8x$, the parabola opens…

(A) Right
(B) Left
(C) Down
(D) Up
Your answer: ____________

4. For $x^2/25 + y^2/9 = 1$, the centre is…

(A) (25, 9)
(B) (0, 0)
(C) (5, 3)
(D) (−5, −3)
Your answer: ____________

5. For $x^2/9 + y^2/25 = 1$, the major axis is…

(A) Vertical
(B) Diagonal
(C) None
(D) Horizontal
Your answer: ____________

6. For $(x - 2)^2/16 + (y + 1)^2/4 = 1$, vertices on the major axis are at…

(A) (2, 3) and (2, −5)
(B) (0, 0) and (16, 4)
(C) (2, −1) and (2, 3)

(D) (−2, −1) and (6, −1)

Your answer: ____________

7. What is the conic of $x^2 + y^2 = 36$?

(A) Ellipse
(B) Circle
(C) Hyperbola
(D) Parabola

Your answer: ____________

8. Why does a satellite dish use a parabolic shape?

(A) It is cheap to manufacture
(B) Ellipses are too round
(C) Parallel signals reflect to a single focus
(D) It looks circular

Your answer: ____________

DID YOU KNOW?

Johannes Kepler discovered around 1609 that planets orbit the sun in ellipses — not circles, as everyone had believed for 2,000 years. He used Tycho Brahe's precise observations of Mars, which a circle simply could not explain. The sun sits at one focus of the orbital ellipse, not at the centre. This single discovery, published in his "New Astronomy", overthrew the ancient view of the heavens and laid the foundation for Newton's laws of motion 70 years later.

UNIT 5 · ANALYTIC GEOMETRY

WEEK 23

Hyperbolas

The fourth and most exotic conic — two separate curves approaching asymptotic lines but never touching. Used in GPS, LORAN navigation, and the orbits of comets that escape the solar system.

Learning Objectives

By the end of this week, you will be able to:

1. Define a hyperbola as a conic section.
2. Identify key features of a hyperbola.
3. Write equations of hyperbolas in standard form.
4. Graph basic hyperbolas using key points and asymptotes.
5. Distinguish between horizontal and vertical hyperbolas.
6. Determine asymptote equations.
7. Apply hyperbolas to real-world contexts.
8. Compare hyperbolas with other conic sections.

Key Concepts

1. What is a hyperbola?

A hyperbola is the set of all points where the absolute difference of distances to two fixed points (foci) is constant. It has two branches that approach asymptotes but never touch them.

2. Standard forms

A hyperbola has two standard forms: horizontal (branches open left-right) and vertical (branches open up-down). The minus sign in the equation is what makes it a hyperbola.

3. Asymptotes — the secret to graphing

A hyperbola has two diagonal asymptote lines through its centre. The branches hug these asymptotes as they extend outward. Draw the asymptotes first, then sketch the curve.

4. Hyperbolas in the real world

Hyperbolas appear in GPS positioning, LORAN long-range navigation, the orbits of escaping comets, and the shock-wave cone of supersonic flight.

5. Worked examples

Five worked hyperbola examples.

Worked Examples

Example 1. Identify the orientation: $(x - 1)^2/4 - (y + 3)^2/9 = 1$.

Solution: Horizontal hyperbola.

Example 2. Find the centre of $(x - 5)^2/16 - (y - 2)^2/9 = 1$.

Solution: $(5, 2)$.

Example 3. Find the vertices of $x^2/9 - y^2/16 = 1$.

Solution: $a = 3$, horizontal. Vertices: $(\pm 3, 0)$.

UNIT 5 · ANALYTIC GEOMETRY

Week 23 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Identify the orientation: $(x - 1)^2/4 - (y + 3)^2/9 = 1$.

2. Find the centre of $(x - 5)^2/16 - (y - 2)^2/9 = 1$.

3. Find the vertices of $x^2/9 - y^2/16 = 1$.

4. Find the asymptotes of $x^2/49 - y^2/16 = 1$.

5. Find the asymptotes of $y^2/25 - x^2/4 = 1$.

6. Identify $(y - 1)^2/9 - (x + 2)^2/4 = 1$.

7. Find vertices of $(x - 3)^2/25 - (y + 1)^2/9 = 1$.

8. Find asymptotes of $(x - 3)^2/25 - (y + 1)^2/9 = 1$.

9. Distinguish: $x^2/9 + y^2/4 = 1$ vs $x^2/9 - y^2/4 = 1$.

10. Write the equation of a horizontal hyperbola with centre (0,0), $a = 4$, $b = 3$.

11. Write the equation of a vertical hyperbola with centre (1, 2), $a = 5$, $b = 3$.

12. A hyperbola has vertices (±6, 0) and asymptotes $y = \pm(1/2)x$. Find a, b, and equation.

13. Why is a hyperbola useful in GPS?

14. What conic models the orbit of an escaping comet?

15. Compare ellipse vs hyperbola: name the key sign difference.

UNIT 5 · ANALYTIC GEOMETRY

Week 23 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. A hyperbola is defined as the set of points where the…

(A) Distance to a focus equals distance to a line
(B) Difference of distances to two foci is constant
(C) Distance from a centre is constant
(D) Sum of distances to two foci is constant
Your answer: ____________

2. For $x^2/16 - y^2/25 = 1$, the orientation is…

(A) Horizontal
(B) Diagonal
(C) Vertical
(D) Cannot be determined
Your answer: ____________

3. For $(x - 3)^2/9 - (y + 1)^2/4 = 1$, the centre is…

(A) $(3, -1)$
(B) $(9, 4)$
(C) $(0, 0)$
(D) $(-3, 1)$
Your answer: ____________

4. For $x^2/16 - y^2/9 = 1$, the vertices are at…

(A) $(\pm 4, 0)$
(B) $(0, \pm 4)$
(C) $(\pm 3, 0)$
(D) $(\pm 16, 0)$
Your answer: ____________

5. For $x^2/25 - y^2/9 = 1$, the asymptotes are…

(A) $y = \pm 3x$
(B) $y = \pm 5x$
(C) $y = \pm(3/5)x$
(D) $y = \pm(5/3)x$
Your answer: ____________

6. For $y^2/4 - x^2/9 = 1$, the asymptotes are…

(A) $y = \pm 2x$
(B) $y = \pm 3x$
(C) $y = \pm(2/3)x$

(D) $y = \pm(3/2)x$

Your answer: ____________

7. A hyperbola differs from an ellipse by having…

(A) No squared terms
(B) One squared term
(C) A minus sign in the equation
(D) A plus sign

Your answer: ____________

8. Why are hyperbolas used in GPS positioning?

(A) GPS satellites orbit hyperbolically
(B) Signal-time differences trace hyperbolas
(C) Earth is hyperbolic
(D) They are easy to draw

Your answer: ____________

DID YOU KNOW?

Halley's Comet has an elliptical orbit and returns every 76 years. But there are also "interstellar" objects — like 'Oumuamua, discovered in 2017, which travelled into our solar system from another star, swung around the sun on a hyperbolic trajectory, and left forever. Its path was indistinguishably hyperbolic, with eccentricity 1.20 — well above the value of 1 that separates open hyperbolic orbits from closed elliptical ones. The maths you are learning this week is what astronomers used to confirm 'Oumuamua was not from our solar system.

UNIT 5 · ANALYTIC GEOMETRY

WEEK 24

Distance, Midpoint, and Slope Formulas

Three foundational coordinate-geometry tools, derived from the Pythagorean theorem. Distance measures length, midpoint locates centres, and slope captures steepness — together they connect algebra to geometry.

Learning Objectives

By the end of this week, you will be able to:

1. Use the distance formula to find lengths between points.
2. Use the midpoint formula to find centres of segments.
3. Calculate slope between two points.
4. Interpret slope as rate of change.
5. Identify parallel and perpendicular lines.
6. Apply formulas in real-world contexts.
7. Solve coordinate geometry problems accurately.
8. Analyse geometric relationships in the coordinate plane.

Key Concepts

1. The distance formula

The distance between two points on the coordinate plane equals the square root of the sum of the squares of the horizontal and vertical differences. It is the Pythagorean theorem in disguise.

2. The midpoint formula

The midpoint of a segment is the average of the endpoints — average the x coordinates and average the y coordinates.

3. The slope formula

Slope is rise over run — the vertical change divided by the horizontal change. It measures the steepness and direction of a line.

4. Parallel and perpendicular

Parallel lines have equal slopes. Perpendicular lines have slopes that are negative reciprocals of each other — multiply to give negative one.

5. Worked examples

Five worked coordinate geometry examples.

Worked Examples

Example 1. Find distance between (2,3) and (5,7).

Solution: $d = \sqrt{[9 + 16]} = \sqrt{25} = 5$.

Example 2. Find distance between (−1,4) and (3,−2).

Solution: $d = \sqrt{[16 + 36]} = \sqrt{52} = 2\sqrt{13} \approx 7.21$.

Example 3. Find distance between (0,0) and (8,15).

Solution: $d = \sqrt{[64 + 225]} = \sqrt{289} = 17$.

UNIT 5 · ANALYTIC GEOMETRY

Week 24 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Find distance between (2,3) and (5,7).

2. Find distance between (−1,4) and (3,−2).

3. Find distance between (0,0) and (8,15).

4. Find midpoint of (6,8) and (10,12).

5. Find midpoint of (−3, 5) and (7, −1).

6. Find slope between (3,5) and (7,9).

7. Find slope between (2,6) and (2,10).

8. Find slope between $(-1,3)$ and $(5,3)$.

9. A line passes through $(0, 0)$ with slope $2/3$. Find another point.

10. Are lines with slopes 2 and $1/2$ perpendicular?

11. Find the slope of a line parallel to $y = 3x + 5$.

12. Find the slope perpendicular to $y = -2x + 1$.

13. Two cities are at $(1,2)$ and $(7,10)$. Distance between?

14. Midpoint between the two cities?

15. A line through $(2, 3)$ is perpendicular to $y = (1/2)x + 4$. Find its slope and write its equation.

UNIT 5 · ANALYTIC GEOMETRY

Week 24 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. Distance between (1, 2) and (4, 6)?

- **(A)** 7
- **(B)** 25
- **(C)** 5
- **(D)** $\sqrt{5}$

Your answer: ____________

2. Distance between (0, 0) and (3, 4)?

- **(A)** 7
- **(B)** 12
- **(C)** 5
- **(D)** 25

Your answer: ____________

3. Midpoint of (2, 4) and (6, 10)?

- **(A)** (2, 6)
- **(B)** (4, 7)
- **(C)** (4, 6)
- **(D)** (8, 14)

Your answer: ____________

4. Midpoint of (−2, 3) and (4, −5)?

- **(A)** (6, −8)
- **(B)** (1, −1)
- **(C)** (2, −2)
- **(D)** (−1, 1)

Your answer: ____________

5. Slope between (1, 2) and (3, 8)?

- **(A)** 1/3
- **(B)** 3
- **(C)** 6
- **(D)** 2

Your answer: ____________

6. Slope between (2, 5) and (2, 9)?

- **(A)** 2
- **(B)** 0
- **(C)** Undefined (vertical)

(D) 4

Your answer: ____________

7. A line has slope 4. The slope of a line perpendicular to it is…

(A) 1/4
(B) 4
(C) −4
(D) −1/4

Your answer: ____________

8. Two lines both have slope −2. They are…

(A) Skew
(B) The same line
(C) Perpendicular
(D) Parallel

Your answer: ____________

DID YOU KNOW?

The coordinate plane was invented by René Descartes around 1637 — and it changed mathematics forever. By labelling every point in space with two numbers, he made it possible to describe geometric shapes with algebraic equations and vice versa. The story goes that Descartes was lying in bed watching a fly crawl on the ceiling, and realised he could describe its position using its distance from two perpendicular walls. From a fly on a ceiling came analytic geometry, calculus, physics, computer graphics, and GPS — all of modern science traces back to that single insight.

UNIT 5 · ANALYTIC GEOMETRY

WEEK 25

Conic Sections Applications

The capstone of Unit 5. Bring together circles, parabolas, ellipses, and hyperbolas, and learn to choose the right model for real-world systems: satellite dishes, planetary orbits, GPS, whispering galleries, and architectural design.

Learning Objectives

By the end of this week, you will be able to:

1. Identify real-world applications of conic sections.
2. Distinguish between parabolic, elliptical, and hyperbolic models.
3. Apply parabolas to reflective and structural systems.
4. Explain elliptical orbits in astronomy.
5. Describe hyperbolas in navigation and signal systems.
6. Select appropriate conic models for scenarios.
7. Interpret graphs of conic sections in context.
8. Solve applied problems involving conic sections.

Key Concepts

1. A tour of the four conics

The four conic sections — circles, ellipses, parabolas, and hyperbolas — appear everywhere in nature, engineering, and astronomy. Each one corresponds to a specific physical behaviour.

2. The parabolic reflection property

Every parabola has the magical property that parallel rays striking its surface all reflect to a single point — the focus. This makes parabolas perfect for satellite dishes, headlights, and telescope mirrors.

3. Ellipses in orbits and acoustics

Planets orbit the sun in ellipses — Kepler's first law. Whispering galleries use the reflection property of ellipses: sound from one focus reflects to the other.

4. Hyperbolas in navigation and physics

Hyperbolas appear when something depends on the difference between two measurements. GPS, LORAN, and the orbits of comets that escape the solar system are all hyperbolic.

5. Choosing your model

When faced with a real-world problem, ask: is it about a closed orbit, a focused reflection, a signal difference, or a simple round shape? The answer tells you which conic to use.

Worked Examples

Example 1. What conic is a satellite dish?

Solution: Parabola — parallel signals reflect to focus.

Example 2. What conic is a planet\\u2019s orbit?

Solution: Ellipse — closed orbit with sun at one focus.

Example 3. What conic is used in GPS?

Solution: Hyperbola — based on signal-time differences.

UNIT 5 · ANALYTIC GEOMETRY

Week 25 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. What conic is a satellite dish?

2. What conic is a planet\\u2019s orbit?

3. What conic is used in GPS?

4. What conic is the silhouette of a cooling tower?

5. Why are parabolic reflectors used for headlights?

6. Why does a whispering gallery work?

7. What\\u2019s the difference between an ellipse and a hyperbola, mathematically?

8. A comet enters the solar system with eccentricity 1.5. What kind of orbit?

9. A telescope focuses starlight at one point. What conic is the mirror?

10. A medical procedure focuses ultrasound at kidney stones from outside the body. What conic?

11. A satellite in geostationary orbit follows what shape?

12. Why is a parabolic shape ideal for a solar oven?

13. In LORAN, why do signal time-differences create hyperbolae?

14. List three real systems that use ellipses.

15. A spacecraft\\u2019s trajectory has eccentricity 0.9. Bound or escape?

UNIT 5 · ANALYTIC GEOMETRY

Week 25 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. A satellite dish uses which conic?

(A) Parabola
(B) Ellipse
(C) Circle
(D) Hyperbola
Your answer: ____________

2. Why are planetary orbits elliptical, not circular?

(A) They are easier
(B) Kepler\\u2019s first law — gravity yields ellipses
(C) Planets are slow
(D) Circles are too small
Your answer: ____________

3. GPS uses which conic geometrically?

(A) Circle
(B) Ellipse
(C) Parabola
(D) Hyperbola
Your answer: ____________

4. A whispering gallery is based on which conic\\u2019s property?

(A) Hyperbola
(B) Ellipse
(C) Parabola
(D) Circle
Your answer: ____________

5. A car headlight uses which conic?

(A) Circle
(B) Hyperbola
(C) Parabola
(D) Ellipse
Your answer: ____________

6. A comet escaping the solar system follows what kind of trajectory?

(A) Parabola
(B) Hyperbola
(C) Ellipse

(D) Circle

Your answer: ____________

7. Which conic has asymptotes?

(A) Parabola
(B) Ellipse
(C) Hyperbola
(D) Circle

Your answer: ____________

8. A nuclear cooling tower\\u2019s hourglass cross-section is which conic shape?

(A) Circle
(B) Parabola
(C) Hyperbola
(D) Ellipse

Your answer: ____________

DID YOU KNOW?

The Voyager 1 and 2 spacecraft, launched in 1977, are now travelling on hyperbolic trajectories that will carry them through interstellar space forever. Voyager 1 crossed the heliopause (the boundary of the sun's influence) in 2012 and is now over 24 billion km away — and still sending faint signals back to Earth using a 22.4 W transmitter, less power than a refrigerator lightbulb. NASA tracks these spacecraft using the exact same hyperbolic-orbit mathematics you are learning this week.

ANSWER KEY

Unit 5 · Analytic Geometry

This answer key covers every practice exercise and quiz question from Unit 5. For full step-by-step solutions to randomised practice generators (separate from the worksheet exercises printed here), refer to the BemandaSTEM Precalculus app.

ANSWER KEY

Week 21 — Circles

Practice Exercises

1. **Q.** Find the circumference of a circle with $r = 7$.
   A. $C = 14\pi \approx 43.98$.
2. **Q.** Find the circumference of a circle with $r = 12$.
   A. $C = 24\pi \approx 75.40$.
3. **Q.** Find the area of a circle with $r = 6$.
   A. $A = 36\pi \approx 113.10$.
4. **Q.** Find the area of a circle with $r = 9$.
   A. $A = 81\pi \approx 254.47$.
5. **Q.** Find the diameter of a circle with $r = 15$.
   A. $d = 30$.
6. **Q.** Write equation of circle centred at origin, $r = 5$.
   A. $x^2 + y^2 = 25$.
7. **Q.** Write equation of circle centred at $(3, -2)$, $r = 4$.
   A. $(x - 3)^2 + (y + 2)^2 = 16$.
8. **Q.** Write equation of circle centred at $(0, 4)$, $r = 7$.
   A. $x^2 + (y - 4)^2 = 49$.
9. **Q.** Find centre and radius of $(x - 5)^2 + (y + 1)^2 = 36$.
   A. Centre $(5, -1)$, radius 6.
10. **Q.** Find centre and radius of $x^2 + y^2 = 81$.
    A. Centre $(0, 0)$, radius 9.
11. **Q.** A circular fountain has radius 4 m. Find circumference.
    A. $C = 8\pi \approx 25.13$ m.
12. **Q.** Find area of the same fountain.
    A. $A = 16\pi \approx 50.27\ \text{m}^2$.
13. **Q.** A circular track has circumference 100 m. Find radius.
    A. $r = 100/(2\pi) \approx 15.92$ m.
14. **Q.** A circular field has area $100\pi\ \text{m}^2$. Find radius.
    A. $\pi r^2 = 100\pi$, so $r = 10$ m.
15. **Q.** A circle is tangent to the x-axis at $(3, 0)$ and has radius 4. Find equation.
    A. Centre is directly above tangent point: $(3, 4)$. Equation: $(x - 3)^2 + (y - 4)^2 = 16$.

Quiz Answers

1. **Answer: (D) 14π**
   Reason: $C = 2\pi(7) = 14\pi$.
2. **Answer: (A) 36π**

Reason: $A = \pi(6)^2 = 36\pi$.

3. **Answer: (A) 10**

   Reason: $r = d/2 = 10$.

4. **Answer: (D) $x^2 + y^2 = 25$**

   Reason: $(x - 0)^2 + (y - 0)^2 = 5^2$ gives $x^2 + y^2 = 25$.

5. **Answer: (A) $(x - 3)^2 + (y + 2)^2 = 16$**

   Reason: $(x - h)^2 + (y - k)^2 = r^2$. $h=3$, $k=-2$, $r=4 \rightarrow (x - 3)^2 + (y + 2)^2 = 16$.

6. **Answer: (B) $(-1, 5)$**

   Reason: Centre $= (h, k)$ where $(x - h)^2 + (y - k)^2 = r^2$. Here $(x - (-1))^2 + (y - 5)^2 = 49$, so centre is $(-1, 5)$.

7. **Answer: (C) 5**

   Reason: $r^2 = 25$, so $r = \sqrt{25} = 5$.

8. **Answer: (A) 90°**

   Reason: A tangent is always perpendicular (90°) to the radius at the point of tangency.

ANSWER KEY

Week 22 — Parabolas and Ellipses

Practice Exercises

1. **Q.** Find the vertex of $(x - 3)^2 = 12(y + 2)$.

   A. Vertex $(3, -2)$. Opens upward.

2. **Q.** Find the vertex of $(y + 1)^2 = -8(x - 4)$.

   A. Vertex $(4, -1)$. Opens to the left.

3. **Q.** Rewrite $y = x^2 - 6x + 5$ in vertex form and find vertex.

   A. $y = (x - 3)^2 - 4$. Vertex $(3, -4)$.

4. **Q.** Rewrite $y = -x^2 + 4x + 1$ in vertex form.

   A. $y = -(x - 2)^2 + 5$. Vertex $(2, 5)$. Opens down.

5. **Q.** Centre of $x^2/49 + y^2/16 = 1$?

   A. Centre $(0, 0)$. $a = 7$, $b = 4$. Major axis horizontal.

6. **Q.** Vertices of $x^2/25 + y^2/9 = 1$?

   A. $a = 5$, major axis along x. Vertices: $(\pm 5, 0)$.

7. **Q.** Co-vertices of the same ellipse?

   A. $b = 3$, minor axis along y. Co-vertices: $(0, \pm 3)$.

8. **Q.** Identify $(x - 2)^2/9 + (y + 3)^2/4 = 1$.

   A. Ellipse, centre $(2, -3)$, $a = 3$, $b = 2$, major axis horizontal.

9. **Q.** Identify $x^2 + y^2 = 25$.

   A. Circle, centre $(0, 0)$, radius 5.

10. **Q.** Identify $(y - 1)^2 = 8(x + 2)$.

    A. Parabola, vertex $(-2, 1)$, opens right.

11. **Q.** Why are parabolas used in headlights?

    A. A light source at the focus produces parallel reflected rays — a beam.

12. **Q.** Why are ellipses used to describe planetary orbits?

    A. Kepler\\u2019s first law: planets move in ellipses with the sun at one focus.

13. **Q.** In a whispering gallery, a person at one focus can clearly hear someone at the other focus. Why?

    A. Sound waves reflect off the elliptical wall and converge from one focus to the other.

14. **Q.** A parabola has vertex $(0, 0)$ and focus $(0, 2)$. Find its equation.

    A. $4p = 4 \cdot 2 = 8$. Equation: $x^2 = 8y$.

15. **Q.** An ellipse has centre $(0,0)$, vertices at $(\pm 10, 0)$, and co-vertices at $(0, \pm 6)$. Find its equation.

    A. $a = 10$, $b = 6$. Equation: $x^2/100 + y^2/36 = 1$.

Quiz Answers

1. **Answer: (C) A focus and a directrix**

   Reason: A parabola: distance to one focus = distance to a fixed line (directrix).

2. **Answer: (A) (4, −2)**

   Reason: $(x - h)^2 = 4p(y - k)$ form; vertex is $(h, k) = (4, -2)$.

3. **Answer: (B) Left**

   Reason: y^2 alone → horizontal parabola. $4p = -8 < 0$, so it opens to the left.

4. **Answer: (B) (0, 0)**

   Reason: No shifts — centre is at the origin.

5. **Answer: (A) Vertical**

   Reason: The larger denominator (25) is under y^2, so major axis is vertical.

6. **Answer: (D) (−2, −1) and (6, −1)**

   Reason: Major axis horizontal ($a^2 = 16$). Centre $(2, -1)$, $a = 4$. Vertices: $(2 - 4, -1) = (-2, -1)$ and $(2 + 4, -1) = (6, -1)$.

7. **Answer: (B) Circle**

   Reason: Both squared terms have the same denominator (coefficient 1) — circle, radius 6.

8. **Answer: (C) Parallel signals reflect to a single focus**

   Reason: A parabola has the property that all parallel rays reflect to its focus — perfect for collecting signals at one point.

ANSWER KEY

Week 23 — Hyperbolas

Practice Exercises

1. **Q.** Identify the orientation: $(x-1)^2/4 - (y+3)^2/9 = 1$.
   A. Horizontal hyperbola.
2. **Q.** Find the centre of $(x-5)^2/16 - (y-2)^2/9 = 1$.
   A. (5, 2).
3. **Q.** Find the vertices of $x^2/9 - y^2/16 = 1$.
   A. $a = 3$, horizontal. Vertices: $(\pm 3, 0)$.
4. **Q.** Find the asymptotes of $x^2/49 - y^2/16 = 1$.
   A. $a = 7$, $b = 4$. $y = \pm(4/7)x$.
5. **Q.** Find the asymptotes of $y^2/25 - x^2/4 = 1$.
   A. $a = 5$, $b = 2$. $y = \pm(5/2)x$.
6. **Q.** Identify $(y-1)^2/9 - (x+2)^2/4 = 1$.
   A. Vertical hyperbola, centre $(-2, 1)$, $a = 3$, $b = 2$.
7. **Q.** Find vertices of $(x-3)^2/25 - (y+1)^2/9 = 1$.
   A. $a = 5$, horizontal. Vertices: $(3 \pm 5, -1) = (-2, -1)$ and $(8, -1)$.
8. **Q.** Find asymptotes of $(x-3)^2/25 - (y+1)^2/9 = 1$.
   A. $y + 1 = \pm(3/5)(x-3)$, so $y = \pm(3/5)(x-3) - 1$.
9. **Q.** Distinguish: $x^2/9 + y^2/4 = 1$ vs $x^2/9 - y^2/4 = 1$.
   A. First is an ellipse, second is a hyperbola — the sign decides.
10. **Q.** Write the equation of a horizontal hyperbola with centre (0,0), $a = 4$, $b = 3$.
    A. $x^2/16 - y^2/9 = 1$.
11. **Q.** Write the equation of a vertical hyperbola with centre (1, 2), $a = 5$, $b = 3$.
    A. $(y-2)^2/25 - (x-1)^2/9 = 1$.
12. **Q.** A hyperbola has vertices $(\pm 6, 0)$ and asymptotes $y = \pm(1/2)x$. Find a, b, and equation.
    A. $a = 6$, $b/a = 1/2 \rightarrow b = 3$. Equation: $x^2/36 - y^2/9 = 1$.
13. **Q.** Why is a hyperbola useful in GPS?
    A. GPS measures time differences from multiple satellites, which trace hyperbolic surfaces; intersecting them pinpoints location.
14. **Q.** What conic models the orbit of an escaping comet?
    A. Hyperbola — the comet has too much energy for a closed elliptical orbit.
15. **Q.** Compare ellipse vs hyperbola: name the key sign difference.
    A. Ellipse: $x^2/a^2 + y^2/b^2 = 1$ (plus). Hyperbola: $x^2/a^2 - y^2/b^2 = 1$ (minus).

Quiz Answers

1. **Answer: (B) Difference of distances to two foci is constant**
   Reason: Difference → hyperbola. Sum → ellipse. Equal → parabola. Constant from centre → circle.

2. **Answer: (A) Horizontal**

   Reason: x^2 *has the positive sign — branches open left/right.*

3. **Answer: (A) (3, −1)**

   Reason: $(h, k) = (3, -1)$.

4. **Answer: (A) (±4, 0)**

   Reason: $a^2 = 16 \rightarrow a = 4$. *Horizontal: vertices at* $(\pm 4, 0)$.

5. **Answer: (C) y = ±(3/5)x**

   Reason: Horizontal: $y = \pm(b/a)x = \pm(3/5)x$.

6. **Answer: (C) y = ±(2/3)x**

   Reason: Vertical: $y = \pm(a/b)x = \pm(2/3)x$.

7. **Answer: (C) A minus sign in the equation**

   Reason: Minus sign between squared terms makes it a hyperbola.

8. **Answer: (B) Signal-time differences trace hyperbolas**

   Reason: Differences in signal arrival times locate the receiver on a hyperbolic curve.

ANSWER KEY

Week 24 — Distance, Midpoint, and Slope Formulas

Practice Exercises

1. **Q.** Find distance between (2,3) and (5,7).
   A. $d = \sqrt{9 + 16} = \sqrt{25} = 5$.
2. **Q.** Find distance between (−1,4) and (3,−2).
   A. $d = \sqrt{16 + 36} = \sqrt{52} = 2\sqrt{13} \approx 7.21$.
3. **Q.** Find distance between (0,0) and (8,15).
   A. $d = \sqrt{64 + 225} = \sqrt{289} = 17$.
4. **Q.** Find midpoint of (6,8) and (10,12).
   A. M = (8, 10).
5. **Q.** Find midpoint of (−3, 5) and (7, −1).
   A. M = (2, 2).
6. **Q.** Find slope between (3,5) and (7,9).
   A. $m = (9-5)/(7-3) = 4/4 = 1$.
7. **Q.** Find slope between (2,6) and (2,10).
   A. Vertical line: slope undefined.
8. **Q.** Find slope between (−1,3) and (5,3).
   A. Horizontal line: slope = 0.
9. **Q.** A line passes through (0, 0) with slope 2/3. Find another point.
   A. Many options. (3, 2) works (using rise/run).
10. **Q.** Are lines with slopes 2 and 1/2 perpendicular?
    A. Product = 1, not −1. Not perpendicular.
11. **Q.** Find the slope of a line parallel to $y = 3x + 5$.
    A. Slope = 3.
12. **Q.** Find the slope perpendicular to $y = -2x + 1$.
    A. Negative reciprocal of −2 is 1/2.
13. **Q.** Two cities are at (1,2) and (7,10). Distance between?
    A. $d = \sqrt{36 + 64} = \sqrt{100} = 10$.
14. **Q.** Midpoint between the two cities?
    A. M = (4, 6).
15. **Q.** A line through (2, 3) is perpendicular to $y = (1/2)x + 4$. Find its slope and write its equation.
    A. Perpendicular slope = −2. Equation: $y - 3 = -2(x - 2)$, or $y = -2x + 7$.

Quiz Answers

1. **Answer: (C)** 5
   Reason: $d = \sqrt{9 + 16} = \sqrt{25} = 5$.
2. **Answer: (C)** 5
   Reason: $d = \sqrt{9 + 16} = 5$. (Classic 3-4-5.)

3. **Answer: (B) (4, 7)**

   Reason: $M = ((2+6)/2, (4+10)/2) = (4, 7)$.

4. **Answer: (B) (1, −1)**

   Reason: $M = ((-2+4)/2, (3+(-5))/2) = (1, -1)$.

5. **Answer: (B) 3**

   Reason: $m = (8 - 2)/(3 - 1) = 6/2 = 3$.

6. **Answer: (C) Undefined (vertical)**

   Reason: Same x-coordinate: vertical line, slope undefined (denominator zero).

7. **Answer: (D) −1/4**

   Reason: Perpendicular: negative reciprocal of 4 is −1/4.

8. **Answer: (D) Parallel**

   Reason: Equal slopes → parallel.

ANSWER KEY

Week 25 — Conic Sections Applications

Practice Exercises

1. **Q.** What conic is a satellite dish?
   A. Parabola — parallel signals reflect to focus.
2. **Q.** What conic is a planet\\u2019s orbit?
   A. Ellipse — closed orbit with sun at one focus.
3. **Q.** What conic is used in GPS?
   A. Hyperbola — based on signal-time differences.
4. **Q.** What conic is the silhouette of a cooling tower?
   A. Hyperbola.
5. **Q.** Why are parabolic reflectors used for headlights?
   A. Bulb at focus → light emerges as parallel beam.
6. **Q.** Why does a whispering gallery work?
   A. Elliptical reflection: sound from one focus converges at the other.
7. **Q.** What\\u2019s the difference between an ellipse and a hyperbola, mathematically?
   A. Ellipse: sum of distances to foci is constant. Hyperbola: absolute difference is constant.
8. **Q.** A comet enters the solar system with eccentricity 1.5. What kind of orbit?
   A. Hyperbolic — it will pass once and escape.
9. **Q.** A telescope focuses starlight at one point. What conic is the mirror?
   A. Parabola.
10. **Q.** A medical procedure focuses ultrasound at kidney stones from outside the body. What conic?
    A. Ellipse — sound source at one focus, stone at the other.
11. **Q.** A satellite in geostationary orbit follows what shape?
    A. A circle (a special-case ellipse with eccentricity 0).
12. **Q.** Why is a parabolic shape ideal for a solar oven?
    A. It concentrates parallel sunlight to a single high-temperature focus.
13. **Q.** In LORAN, why do signal time-differences create hyperbolae?
    A. A constant time-difference means a constant distance-difference — the definition of a hyperbola.
14. **Q.** List three real systems that use ellipses.
    A. Planetary orbits, whispering galleries, kidney-stone lithotripsy.
15. **Q.** A spacecraft\\u2019s trajectory has eccentricity 0.9. Bound or escape?
    A. Eccentricity < 1, so still a bound ellipse — but very stretched.

Quiz Answers

1. **Answer: (A) Parabola**
   Reason: Parabolic reflection: parallel signals → single focus.

2. **Answer: (B) Kepler\\u2019s first law — gravity yields ellipses**

   Reason: Newton showed gravity produces elliptical bound orbits with the central body at one focus.

3. **Answer: (D) Hyperbola**

   Reason: Time-difference of signal arrival traces a hyperbolic locus of possible positions.

4. **Answer: (B) Ellipse**

   Reason: Sound from one focus reflects to the other — the ellipse\\u2019s reflection property.

5. **Answer: (C) Parabola**

   Reason: Bulb at the focus → parallel beam emerges.

6. **Answer: (B) Hyperbola**

   Reason: An open path with eccentricity > 1.

7. **Answer: (C) Hyperbola**

   Reason: Only hyperbolas have asymptotic lines that the branches approach.

8. **Answer: (C) Hyperbola**

   Reason: A hyperboloid (3D hyperbola) — structurally strong and aerodynamically efficient.

UNIT 6
Systems and Matrices

UNIT OVERVIEW

Two weeks on the algebra of multiple equations and the language of data tables. Systems of equations and inequalities (graphing, substitution, elimination) followed by matrix operations — the structures used in linear algebra, computer graphics, and scientific computing.

Weeks in this unit:

Week 26 — *Solving Systems of Equations and Inequalities*
Week 27 — *Matrix Operations and Applications*

UNIT 6 · SYSTEMS AND MATRICES

WEEK 26

Solving Systems of Equations and Inequalities

Two equations, two unknowns — or more. Master the three solving methods (graphing, substitution, elimination), classify the three possible outcomes, and graph inequality regions.

Learning Objectives

By the end of this week, you will be able to:

1. Define a system of equations and inequalities.
2. Solve systems using graphing, substitution, and elimination.
3. Identify types of solutions for systems.
4. Graph linear inequalities correctly.
5. Determine solution regions for inequalities.
6. Interpret solutions in real-world contexts.
7. Choose appropriate solving methods.
8. Analyse and verify solutions.

Key Concepts

1. What is a system?

A system of equations is two or more equations with the same variables. The solution is the point (or set of points) that makes all equations true simultaneously.

2. Three solution types

A linear system can have one solution (lines cross), no solution (parallel lines), or infinitely many solutions (same line).

3. Three solving methods

Graphing shows the picture. Substitution solves one equation for a variable and plugs into the other. Elimination adds equations to cancel a variable.

4. Systems of inequalities

For inequalities, the solution is a region — the overlap of all individual shaded regions. Use dashed lines for strict inequality, solid for inclusive.

5. Worked examples

Five worked system examples.

Worked Examples

Example 1. Solve by substitution: $y = x + 3$, $y = 2x + 1$.

Solution: $x + 3 = 2x + 1 \rightarrow x = 2, y = 5$. (2, 5).

Example 2. Solve by elimination: $x + y = 8$, $x - y = 2$.

Solution: Add: $2x = 10 \rightarrow x = 5, y = 3$.

Example 3. Solve by graphing: $y = 3x - 1$, $y = -x + 5$.

Solution: Set equal: $3x - 1 = -x + 5 \rightarrow x = 1.5, y = 3.5$. (1.5, 3.5).

UNIT 6 · SYSTEMS AND MATRICES

Week 26 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Solve by substitution: $y = x + 3$, $y = 2x + 1$.

2. Solve by elimination: $x + y = 8$, $x - y = 2$.

3. Solve by graphing: $y = 3x - 1$, $y = -x + 5$.

4. Classify $y = 2x + 1$, $y = 2x + 5$.

5. Classify $y = x + 3$, $2y = 2x + 6$.

6. Graph $x + y \geq 4$. What kind of line?

7. Graph $y < 2x + 1$. What kind of line?

8. A store sells chairs and tables. 3 chairs + 1 table = $200; 1 chair + 2 tables = $250. Find price of each.

9. For $2x + 3y = 12$ and $x - y = 1$, use substitution.

10. For $4x + y = 9$ and $2x - y = 1$, use elimination.

11. Does (1, 4) satisfy $y \geq 2x + 1$?

12. Does (2, 1) satisfy $y > 2x + 1$?

13. A system models supply ($y = 2x + 1$) and demand ($y = -x + 10$). Find equilibrium.

14. A break-even: $R = 10x$, $C = 4x + 60$. When $R = C$?

15. Why might a system have "no solution" in a real-world context?

UNIT 6 · SYSTEMS AND MATRICES

Week 26 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. The solution to a system of two linear equations is geometrically…

- **(A)** The midpoint of the lines
- **(B)** Any point on either line
- **(C)** The sum of the lines
- **(D)** The intersection of the two lines

Your answer: ____________

2. Two parallel lines form a system with…

- **(A)** One solution
- **(B)** No solution
- **(C)** Cannot say
- **(D)** Infinitely many solutions

Your answer: ____________

3. Solve $y = 2x + 1$ and $y = x + 5$.

- **(A)** (1, 3)
- **(B)** (4, 9)
- **(C)** (2, 5)
- **(D)** (0, 1)

Your answer: ____________

4. Solve $x + y = 8$ and $x - y = 2$ using elimination.

- **(A)** (8, 2)
- **(B)** (2, 6)
- **(C)** (5, 3)
- **(D)** (3, 5)

Your answer: ____________

5. For $y < 3x + 2$, the boundary line is…

- **(A)** vertical
- **(B)** dashed
- **(C)** curved
- **(D)** solid

Your answer: ____________

6. For $2x + 3y = 12$ and $2x - 3y = 6$, which method is most efficient?

- **(A)** Graphing
- **(B)** Matrices
- **(C)** Substitution

(D) Elimination

Your answer: ____________

7. A system y = 2x + 1 and 2y = 4x + 2 has...

(A) Two solutions

(B) Infinitely many solutions

(C) No solution

(D) One solution

Your answer: ____________

8. Revenue R = 8x, cost C = 3x + 25. Break-even point?

(A) 25 units

(B) 5 units

(C) 3 units

(D) 8 units

Your answer: ____________

DID YOU KNOW?

Systems of equations are everywhere in real life. Airlines solve enormous systems daily to schedule crews and aircraft. Power grids solve them every second to balance electricity supply and demand. Even the GPS in your phone solves a system of four equations from four satellites to pinpoint your location anywhere on Earth.

UNIT 6 · SYSTEMS AND MATRICES

WEEK 27

Matrix Operations and Applications

A matrix is a rectangular grid of numbers. Master the four operations (addition, scalar multiplication, matrix multiplication, determinants), and meet the structures that power linear algebra, computer graphics, and machine learning.

Learning Objectives

By the end of this week, you will be able to:

1. Define a matrix and identify its order.
2. Classify different types of matrices.
3. Perform addition and subtraction of matrices.
4. Multiply matrices by scalars.
5. Perform basic matrix multiplication.
6. Understand the concept of determinants and inverses.
7. Apply matrices to real-world problems.
8. Interpret solutions represented in matrix form.

Key Concepts

1. What is a matrix?

A matrix is a rectangular array of numbers organized in rows and columns. Its order is the number of rows by the number of columns.

2. Addition, subtraction, scalar multiplication

Add or subtract matrices of the same order by combining corresponding entries. Multiply by a scalar to scale every entry.

3. Matrix multiplication

Matrix multiplication uses the row-by-column rule. The number of columns in the first must match the number of rows in the second.

4. Determinant and inverse

The determinant of a 2 by 2 matrix is a d minus b c. The inverse, when it exists, undoes the matrix — A times A inverse equals the identity.

5. Worked examples

Five matrix operation examples.

Worked Examples

Example 1. Order of [1 4; 5 6]?

Solution: 2×2.

Example 2. $[1\ 4;\ 5\ 6] + [2\ 3;\ 7\ 8] = ?$

Solution: $[3\ 7;\ 12\ 14]$.

Example 3. $2 \times [3\ 5;\ 1\ 2] = ?$

Solution: $[6\ 10;\ 2\ 4]$.

UNIT 6 · SYSTEMS AND MATRICES

Week 27 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Order of [1 4; 5 6]?

2. [1 4; 5 6] + [2 3; 7 8] = ?

3. 2 × [3 5; 1 2] = ?

4. [1 2; 4 5] · [3; 6] = ?

5. Determinant of [3 7; 2 5]?

6. Determinant of [4 6; 2 3]?

7. Is the identity matrix I its own inverse?

8. Multiply [1 0 0; 0 1 0; 0 0 1] by any 3×1 vector — what happens?

9. For [2 1; 4 2], find determinant. Is it invertible?

10. Compute [2 3] · [1; 4].

11. If A is 3×2, what order matrix can be added to A?

12. If A is 3×2 and B is 2×4, what is the order of AB?

13. Is AB = BA in general?

14. A school stores attendance as a 5×7 matrix (5 days, 7 classes). What does each entry mean?

15. How many entries does a 4 × 6 matrix have?

UNIT 6 · SYSTEMS AND MATRICES

Week 27 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. The order of [1 2 3; 4 5 6] is…

- **(A)** 6 × 1
- **(B)** 2 × 3
- **(C)** 1 × 6
- **(D)** 3 × 2

Your answer: ____________

2. Determinant of [3 7; 2 5]?

- **(A)** 1
- **(B)** −1
- **(C)** 15
- **(D)** 29

Your answer: ____________

3. Determinant of [2 4; 1 2]?

- **(A)** 6
- **(B)** 8
- **(C)** 0
- **(D)** 4

Your answer: ____________

4. If A is 2×3 and B is 3×2, A·B has order…

- **(A)** 3 × 3
- **(B)** 2 × 3
- **(C)** Undefined
- **(D)** 2 × 2

Your answer: ____________

5. For matrix [a b; c d], inverse exists when…

- **(A)** ad − bc ≠ 0
- **(B)** a + d ≠ 0
- **(C)** always
- **(D)** ad + bc ≠ 0

Your answer: ____________

DID YOU KNOW?

Modern computer graphics, including every video game and animated film you have ever seen, runs on matrix multiplication. A 3D rotation is encoded as a 3×3 or 4×4 matrix; rotating a character means multiplying every vertex by that matrix. The original Google PageRank algorithm was a giant matrix computation — multiplying the entire World Wide Web by itself, repeatedly, until it converged to a ranking of every page. Matrices are the language of linear algebra, the language of modern computing.

ANSWER KEY

Unit 6 · Systems and Matrices

This answer key covers every practice exercise and quiz question from Unit 6. For full step-by-step solutions to randomised practice generators (separate from the worksheet exercises printed here), refer to the BemandaSTEM Precalculus app.

ANSWER KEY

Week 26 — Solving Systems of Equations and Inequalities

Practice Exercises

1. **Q.** Solve by substitution: $y = x + 3$, $y = 2x + 1$.

   A. $x + 3 = 2x + 1 \rightarrow x = 2$, $y = 5$. (2, 5).

2. **Q.** Solve by elimination: $x + y = 8$, $x - y = 2$.

   A. Add: $2x = 10 \rightarrow x = 5$, $y = 3$.

3. **Q.** Solve by graphing: $y = 3x - 1$, $y = -x + 5$.

   A. Set equal: $3x - 1 = -x + 5 \rightarrow x = 1.5$, $y = 3.5$. (1.5, 3.5).

4. **Q.** Classify $y = 2x + 1$, $y = 2x + 5$.

   A. Same slope, different intercept → no solution.

5. **Q.** Classify $y = x + 3$, $2y = 2x + 6$.

   A. Second is 2× first → same line → infinite solutions.

6. **Q.** Graph $x + y \geq 4$. What kind of line?

   A. Solid line through (0,4) and (4,0); shade above.

7. **Q.** Graph $y < 2x + 1$. What kind of line?

   A. Dashed line; shade below.

8. **Q.** A store sells chairs and tables. 3 chairs + 1 table = \$200; 1 chair + 2 tables = \$250. Find price of each.

   A. Solve system: chair = 50, table = 100.

9. **Q.** For $2x + 3y = 12$ and $x - y = 1$, use substitution.

   A. $x = y + 1$. Substitute: $2(y+1) + 3y = 12 \rightarrow 5y = 10 \rightarrow y = 2$, $x = 3$.

10. **Q.** For $4x + y = 9$ and $2x - y = 1$, use elimination.

    A. Add: $6x = 10 \rightarrow x \approx 1.67$, $y \approx 2.33$.

11. **Q.** Does (1, 4) satisfy $y \geq 2x + 1$?

    A. $4 \geq 3$ ✓. Yes.

12. **Q.** Does (2, 1) satisfy $y > 2x + 1$?

    A. $1 > 5$? No.

13. **Q.** A system models supply ($y = 2x + 1$) and demand ($y = -x + 10$). Find equilibrium.

    A. $2x + 1 = -x + 10 \rightarrow 3x = 9 \rightarrow x = 3$, $y = 7$.

14. **Q.** A break-even: $R = 10x$, $C = 4x + 60$. When $R = C$?

    A. $10x = 4x + 60 \rightarrow 6x = 60 \rightarrow x = 10$.

15. **Q.** Why might a system have "no solution" in a real-world context?

    A. The two conditions are contradictory — no single combination of variables satisfies both.

Quiz Answers

1. **Answer: (D) The intersection of the two lines**

   Reason: A solution makes both equations true — graphically, this is the intersection.

2. **Answer: (B) No solution**

Reason: Parallel lines never meet — the system is inconsistent.

3. **Answer: (B) (4, 9)**

   Reason: $2x + 1 = x + 5 \rightarrow x = 4 \rightarrow y = 9$.

4. **Answer: (C) (5, 3)**

   Reason: Add: $2x = 10 \rightarrow x = 5, y = 3$.

5. **Answer: (B) dashed**

   Reason: Strict inequality $<$ $\rightarrow$ *dashed line.*

6. **Answer: (D) Elimination**

   Reason: y *coefficients are opposites* $\rightarrow$ *add equations to eliminate* y.

7. **Answer: (B) Infinitely many solutions**

   Reason: Second equation is 2× the first — same line.

8. **Answer: (B) 5 units**

   Reason: $8x = 3x + 25 \rightarrow 5x = 25 \rightarrow x = 5$.

ANSWER KEY

Week 27 — Matrix Operations and Applications

Practice Exercises

1. **Q.** Order of [1 4; 5 6]?

   A. 2 × 2.

2. **Q.** [1 4; 5 6] + [2 3; 7 8] = ?

   A. [3 7; 12 14].

3. **Q.** 2 × [3 5; 1 2] = ?

   A. [6 10; 2 4].

4. **Q.** [1 2; 4 5] · [3; 6] = ?

   A. [15; 42].

5. **Q.** Determinant of [3 7; 2 5]?

   A. 15 − 14 = 1.

6. **Q.** Determinant of [4 6; 2 3]?

   A. 12 − 12 = 0. Singular.

7. **Q.** Is the identity matrix I its own inverse?

   A. Yes: I · I = I.

8. **Q.** Multiply [1 0 0; 0 1 0; 0 0 1] by any 3×1 vector — what happens?

   A. The vector is unchanged. (Identity matrix preserves vectors.)

9. **Q.** For [2 1; 4 2], find determinant. Is it invertible?

   A. 4 − 4 = 0. Singular — no inverse.

10. **Q.** Compute [2 3] · [1; 4].

    A. 2·1 + 3·4 = 14. (A 1×2 times a 2×1 gives a 1×1 scalar.)

11. **Q.** If A is 3×2, what order matrix can be added to A?

    A. Any 3×2 matrix.

12. **Q.** If A is 3×2 and B is 2×4, what is the order of AB?

    A. 3 × 4.

13. **Q.** Is AB = BA in general?

    A. No, matrix multiplication is not commutative.

14. **Q.** A school stores attendance as a 5×7 matrix (5 days, 7 classes). What does each entry mean?

    A. Number of students present in class j on day i.

15. **Q.** How many entries does a 4 × 6 matrix have?

    A. 24.

Quiz Answers

1. **Answer: (B) 2 × 3**

   Reason: 2 rows and 3 columns → order 2 × 3.

2. **Answer: (A) 1**

   Reason: $|A| = 3 \cdot 5 - 7 \cdot 2 = 15 - 14 = 1$.

3. **Answer: (C)** $\mathbf{0}$

   Reason: $2{\cdot}2 - 4{\cdot}1 = 0$. *(Rows are proportional → singular.)*

4. **Answer: (D)** $\mathbf{2 \times 2}$

   Reason: $(m \times n)(n \times p) = m \times p$, *so* 2×2.

5. **Answer: (A)** $\mathbf{ad - bc \neq 0}$

   Reason: Inverse exists iff determinant $\neq 0$.

UNIT 7
Sequences and Series

UNIT OVERVIEW

Patterns of numbers and their sums. Master arithmetic and geometric sequences, learn to express them with explicit and recursive formulas, and compute series with sigma notation — the building blocks of compound interest, population models, and calculus.

Weeks in this unit:

UNIT 7 · SEQUENCES AND SERIES

WEEK 28

Arithmetic Sequences

A list of numbers where each step adds the same amount. Master the explicit and recursive formulas, find any term without writing them all out, and model linear-growth situations.

Learning Objectives

By the end of this week, you will be able to:

1. Define an arithmetic sequence.
2. Identify the common difference.
3. Write the explicit formula for an arithmetic sequence.
4. Find the nth term of a sequence.
5. Use recursive reasoning for sequences.
6. Solve basic sequence problems.
7. Apply arithmetic sequences to real-world contexts.
8. Interpret patterns in numerical sequences.

Key Concepts

1. What is a sequence?

A sequence is an ordered list of numbers following a pattern. An arithmetic sequence adds the same constant to get from one term to the next.

2. Explicit formula

The nth term equals the first term plus n minus one times the common difference.

3. Recursive formula

A recursive formula defines each term using the previous one. It tells you how to step from one term to the next.

4. Sum of an arithmetic sequence

The sum of the first n terms equals n over two times the sum of the first and last terms.

5. Worked examples

Five worked arithmetic sequence examples.

Worked Examples

Example 1. Find d: 3, 6, 9, 12.

Solution: $d = 3$.

Example 2. Find d: 20, 15, 10, 5.

Solution: $d = -5$.

Example 3. $a_1 = 2$, $d = 5$. Find a_6.

Solution: $2 + 5 \cdot 5 = 27$.

UNIT 7 · SEQUENCES AND SERIES

Week 28 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Find d: 3, 6, 9, 12.

2. Find d: 20, 15, 10, 5.

3. $a_1 = 2$, $d = 5$. Find a_6.

4. $a_1 = 10$, $d = -2$. Find a_{10}.

5. $a_1 = 4$, $a_7 = 22$. Find d.

6. Find sum $1+3+5+\ldots+99$.

7. Find sum $2+5+8+\ldots+50$.

8. Write $a_1 = 5$; $d = 4$ in recursive form.

9. Is 1, 2, 4, 8 arithmetic?

10. A theatre has 20 seats in row 1; each row has 2 more seats. How many seats in row 15?

11. Total seats in rows 1–15?

12. Find a_n if $a_1 = 7$, $d = -3$, and $a_n = -20$.

13. Are 2, 4, 8, 16 arithmetic?

14. Find common difference: −5, −2, 1, 4.

15. Sum of first 20 terms of 2, 5, 8, 11, …?

UNIT 7 · SEQUENCES AND SERIES

Week 28 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. In an arithmetic sequence, the difference between consecutive terms is…

(A) Constant
(B) Increasing
(C) Always positive
(D) A ratio
Your answer: ____________

2. Find d in 3, 8, 13, 18, …

(A) −5
(B) 3
(C) 5
(D) 15
Your answer: ____________

3. For $a_1 = 2$, $d = 4$, find a_5.

(A) 20
(B) 22
(C) 18
(D) 16
Your answer: ____________

4. Classify 5, 10, 15, 20.

(A) Both
(B) Geometric
(C) Neither
(D) Arithmetic
Your answer: ____________

5. $a_1 = 3$, $a_{10} = 30$. Find d.

(A) 10
(B) 27
(C) 9
(D) 3
Your answer: ____________

6. Sum 1+2+3+…+100.

(A) 5050
(B) 100
(C) 550

(D) 10000

Your answer: ____________

7. Recursive form for $a_1 = 4$, $d = 5$?

(A) $a_1 = 4$; $a_n = a_{n-1} + 5$
(B) $a_n = 4 + 5n$
(C) $a_n = 4 \cdot 5^n$
(D) $a_n = 4^n$

Your answer: ____________

8. Find a_{20} for $a_1 = 1$, $d = 2$.

(A) 41
(B) 40
(C) 21
(D) 39

Your answer: ____________

DID YOU KNOW?

When Carl Friedrich Gauss was eight years old, his teacher tried to keep the class busy by asking them to add all integers from 1 to 100. Gauss solved it in seconds by noticing the pairs (1+100, 2+99, ..., 50+51) all summed to 101 — giving 50 × 101 = 5050. That insight, made by a child, became the formula for any arithmetic series.

UNIT 7 · SEQUENCES AND SERIES

WEEK 29

Geometric Sequences

Sequences that multiply instead of add. Master the common ratio, the explicit formula, and connect geometric sequences to exponential growth — the secret behind compound interest and viral spread.

Learning Objectives

By the end of this week, you will be able to:

1. Define a geometric sequence.
2. Identify the common ratio.
3. Write the explicit formula for a geometric sequence.
4. Find the nth term of a sequence.
5. Use recursive reasoning for sequences.
6. Solve basic geometric sequence problems.
7. Apply geometric sequences to real-world contexts.
8. Distinguish between arithmetic and geometric sequences.

Key Concepts

1. Multiplying not adding

A geometric sequence is one in which each term is found by multiplying the previous term by a constant — the common ratio.

2. Explicit formula

The nth term equals the first term times r raised to the n minus one power.

3. Geometric series sum

The sum of the first n terms of a geometric sequence is a one times the quantity one minus r to the n, divided by one minus r.

4. Arithmetic vs geometric

Arithmetic sequences add the same number; geometric sequences multiply by the same number. Arithmetic builds linearly; geometric explodes (or shrinks) exponentially.

5. Worked examples

Five worked geometric examples.

Worked Examples

Example 1. Find r: 3, 6, 12, 24.

Solution: $r = 2$.

Example 2. Find r: 81, 27, 9, 3.

Solution: $r = 1/3$.

Example 3. $a_1 = 2$, $r = 3$. Find a_5.

Solution: $2 \cdot 3^4 = 2 \cdot 81 = 162$.

UNIT 7 · SEQUENCES AND SERIES

Week 29 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Find r: 3, 6, 12, 24.

2. Find r: 81, 27, 9, 3.

3. $a_1 = 2$, $r = 3$. Find a_5.

4. $a_1 = 100$, $r = 0.5$. Find a_6.

5. Classify 1, 4, 16, 64.

6. Sum first 4 terms of 1, 3, 9, 27.

7. Sum first 5 terms of 2, 6, 18, 54, 162.

8. Infinite sum of 1 + 1/2 + 1/4 + 1/8 + …

9. Bacteria triples every hour. Start with 5. Population after 4 hours?

10. $500 invested at 5% compound annually for 10 years?

11. Half-life decay: 80 g halves every hour. Mass after 5 hours?

12. A geometric sequence has $a_1 = 4$, $a_5 = 324$. Find r.

13. Is 1, 4, 9, 16 arithmetic, geometric, or neither?

14. Sum 1 + 2 + 4 + 8 + … + 1024.

15. A car worth $20,000 depreciates 15% per year. Value after 5 years?

UNIT 7 · SEQUENCES AND SERIES

Week 29 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. In a geometric sequence, each term is found by…

(A) Squaring
(B) Adding d
(C) Multiplying by r
(D) Doubling
Your answer: ____________

2. Find r in 4, 8, 16, 32, …

(A) 12
(B) 4
(C) 2
(D) 8
Your answer: ____________

3. For $a_1 = 3$, $r = 4$, find a_5.

(A) 15
(B) 384
(C) 768
(D) 12
Your answer: ____________

4. Classify 2, 4, 6, 8.

(A) Geometric
(B) Arithmetic
(C) Both
(D) Neither
Your answer: ____________

5. Classify 5, 10, 20, 40.

(A) Neither
(B) Both
(C) Arithmetic
(D) Geometric
Your answer: ____________

6. Sum 1+2+4+8+16.

(A) 31
(B) 15
(C) 30

(D) 32

Your answer: ____________

7. A bacteria culture starts at 100 and doubles every hour. Population after 4 hours?

(A) 400
(B) 3200
(C) 1600
(D) 800

Your answer: ____________

8. $1000 invested at 6% compounded annually for 10 years (approximate)?

(A) $1060
(B) ≈ $1791
(C) $10000
(D) $1600

Your answer: ____________

DID YOU KNOW?

A folded piece of paper is the most striking example of a geometric sequence. Fold a sheet 7 times and it is the thickness of a notebook. Fold it 23 times (if you could) and it would reach a kilometer high. Fold it 42 times and it would reach the Moon. Each fold doubles the thickness — geometric sequence with $r = 2$ grows astonishingly fast.

UNIT 7 · SEQUENCES AND SERIES

WEEK 30

Summation Formulas and Applications

Sigma notation gives a compact language for sums of any length. Master the properties, apply the series formulas, and connect summation to real-world totals — wages, savings, populations.

Learning Objectives

By the end of this week, you will be able to:

1. Interpret sigma (Σ) notation correctly.
2. Evaluate basic summations.
3. Apply summation properties.
4. Use formulas for arithmetic series.
5. Use formulas for geometric series.
6. Solve real-world summation problems.
7. Break complex sums into simpler parts.
8. Interpret results in context.

Key Concepts

1. Sigma notation

Sigma notation uses the Greek capital sigma to write long sums compactly. The variable goes underneath, the limit above.

2. Summation properties

Summation distributes over addition and pulls out constants. Both behave like the linearity rule of derivatives or integrals.

3. Arithmetic and geometric series formulas

For an arithmetic series, the sum is n over two times the sum of the first and last terms. For a geometric series, use the geometric sum formula.

4. Breaking complex sums

When a summation has multiple parts, split it using linearity, evaluate each piece separately, then combine.

5. Worked examples

Five worked summation examples.

Worked Examples

Example 1. Evaluate $\sum_{i=1}^{6} i$.

Solution: $6 \cdot 7/2 = 21$.

Example 2. Evaluate $\sum_{i=1}^{10} i$.

Solution: $10 \cdot 11/2 = 55$.

Example 3. Evaluate $\sum_{i=1}^{5} 4$.

Solution: $5 \cdot 4 = 20$.

UNIT 7 · SEQUENCES AND SERIES

Week 30 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Evaluate $\sum_{i=1}^{6} i$.

2. Evaluate $\sum_{i=1}^{10} i$.

3. Evaluate $\sum_{i=1}^{5} 4$.

4. Sum $3+6+9+12+15$.

5. Geometric sum $5+10+20+40$.

6. $\sum_{i=1}^{3} (i + 2) = ?$

7. $\sum_{i=1}^{4} (2i - 1) = ?$

8. Sum of infinite series 1 + 0.1 + 0.01 + 0.001 + …

9. Sum of infinite series 4 + 2 + 1 + 0.5 + …

10. Save $20 first week, +$5 each week. Total after 8 weeks?

11. A theatre has 30 seats in row 1, +3 per row. Total in 12 rows?

12. Bouncing ball: drop from 10 m, bounces 60% each time. Total distance?

13. Express 1+2+3+…+100 in sigma notation.

14. Express 2+4+6+…+100 in sigma notation.

15. Repeating decimal 0.999… as an infinite geometric series equals…

UNIT 7 · SEQUENCES AND SERIES

Week 30 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. $\sum_{i=1}^{4} i = ?$

- **(A)** 4
- **(B)** 16
- **(C)** 24
- **(D)** 10

Your answer: ____________

2. $\sum_{i=1}^{6} i = ?$

- **(A)** 36
- **(B)** 21
- **(C)** 6
- **(D)** 15

Your answer: ____________

3. $\sum_{i=1}^{5} 3 = ?$

- **(A)** 8
- **(B)** 15
- **(C)** 5
- **(D)** 3

Your answer: ____________

4. Sum of 1+3+5+7+9?

- **(A)** 25
- **(B)** 45
- **(C)** 15
- **(D)** 20

Your answer: ____________

5. Geometric sum 1+3+9+27?

- **(A)** 13
- **(B)** 15
- **(C)** 81
- **(D)** 40

Your answer: ____________

6. $\sum_{i=1}^{3} (2i+1) = ?$

- **(A)** 3
- **(B)** 21
- **(C)** 15

(D) 9

Your answer: ____________

7. Infinite sum of 1+1/2+1/4+1/8+…

(A) 3/2

(B) 2

(C) 1

(D) ∞

Your answer: ____________

8. Save $5 first week, +$3 weekly. Total after 10 weeks?

(A) $185

(B) $95

(C) $50

(D) $30

Your answer: ____________

DID YOU KNOW?

The capital Greek letter sigma was first used for summation by Leonhard Euler in 1755. He chose it as the Greek equivalent of "S" for "sum." That single notation has been used in virtually every mathematics paper since — algorithms, statistics, physics equations, machine learning loss functions all rely on the same compact symbol Euler picked nearly 300 years ago.

ANSWER KEY

Unit 7 · Sequences and Series

This answer key covers every practice exercise and quiz question from Unit 7. For full step-by-step solutions to randomised practice generators (separate from the worksheet exercises printed here), refer to the BemandaSTEM Precalculus app.

ANSWER KEY

Week 28 — Arithmetic Sequences

Practice Exercises

1. **Q.** Find d: 3, 6, 9, 12.
   A. $d = 3$.
2. **Q.** Find d: 20, 15, 10, 5.
   A. $d = -5$.
3. **Q.** $a_1 = 2$, $d = 5$. Find a_6.
   A. $2 + 5 \cdot 5 = 27$.
4. **Q.** $a_1 = 10$, $d = -2$. Find a_{10}.
   A. $10 + 9(-2) = -8$.
5. **Q.** $a_1 = 4$, $a_7 = 22$. Find d.
   A. $22 = 4 + 6d \rightarrow d = 3$.
6. **Q.** Find sum 1+3+5+…+99.
   A. $a_1 = 1$, $a_n = 99$, $n = 50$. $S = 50/2 \cdot 100 = 2500$.
7. **Q.** Find sum 2+5+8+…+50.
   A. $a_1 = 2$, $d = 3$, $a_n = 50 \rightarrow n = 17$. $S = 17/2 \cdot 52 = 442$.
8. **Q.** Write $a_1 = 5$; $d = 4$ in recursive form.
   A. $a_1 = 5$; $a_n = a_{n-1} + 4$.
9. **Q.** Is 1, 2, 4, 8 arithmetic?
   A. No — differences are 1, 2, 4 (not constant). Geometric (ratio 2).
10. **Q.** A theatre has 20 seats in row 1; each row has 2 more seats. How many seats in row 15?
    A. $20 + 14 \cdot 2 = 48$.
11. **Q.** Total seats in rows 1–15?
    A. $a_1 = 20$, $a_{15} = 48$, $n = 15$. $S = 15/2 \cdot 68 = 510$.
12. **Q.** Find a_n if $a_1 = 7$, $d = -3$, and $a_n = -20$.
    A. $-20 = 7 + (n-1)(-3) \rightarrow -27 = -3(n-1) \rightarrow n = 10$.
13. **Q.** Are 2, 4, 8, 16 arithmetic?
    A. No — differences are 2, 4, 8.
14. **Q.** Find common difference: −5, −2, 1, 4.
    A. $d = 3$.
15. **Q.** Sum of first 20 terms of 2, 5, 8, 11, …?
    A. $a_1 = 2$, $d = 3$, $a_{20} = 59$. $S = 20/2 \cdot 61 = 610$.

Quiz Answers

1. **Answer: (A) Constant**
   Reason: Constant common difference d defines arithmetic.
2. **Answer: (C) 5**
   Reason: $8 - 3 = 5$.

3. **Answer: (C) 18**

   Reason: $a_5 = 2 + (5-1)\cdot 4 = 2 + 16 = 18$.

4. **Answer: (D) Arithmetic**

   Reason: Common difference 5 → arithmetic.

5. **Answer: (D) 3**

   Reason: $30 = 3 + 9d \rightarrow 9d = 27 \rightarrow d = 3$.

6. **Answer: (A) 5050**

   Reason: $S = 100/2 \cdot 101 = 5050$.

7. **Answer: (A)** $a_1 = 4; a_n = a_{n-1} + 5$

   Reason: Recursive: each term equals previous plus d.

8. **Answer: (D) 39**

   Reason: $1 + 19\cdot 2 = 39$.

ANSWER KEY

Week 29 — Geometric Sequences

Practice Exercises

1. **Q.** Find r: 3, 6, 12, 24.
   A. $r = 2$.
2. **Q.** Find r: 81, 27, 9, 3.
   A. $r = 1/3$.
3. **Q.** $a_1 = 2$, $r = 3$. Find a_5.
   A. $2 \cdot 3^4 = 2 \cdot 81 = 162$.
4. **Q.** $a_1 = 100$, $r = 0.5$. Find a_6.
   A. $100 \cdot 0.5^5 = 100 \cdot 0.03125 = 3.125$.
5. **Q.** Classify 1, 4, 16, 64.
   A. Geometric, $r = 4$.
6. **Q.** Sum first 4 terms of 1, 3, 9, 27.
   A. $S_4 = 1(1 - 81)/(1 - 3) = 40$.
7. **Q.** Sum first 5 terms of 2, 6, 18, 54, 162.
   A. $S_5 = 2(1 - 243)/(1 - 3) = 242$.
8. **Q.** Infinite sum of $1 + 1/2 + 1/4 + 1/8 + \ldots$
   A. $S_\infty = 1/(1 - 1/2) = 2$.
9. **Q.** Bacteria triples every hour. Start with 5. Population after 4 hours?
   A. $5 \cdot 3^4 = 5 \cdot 81 = 405$.
10. **Q.** $500 invested at 5% compound annually for 10 years?
    A. $500 \cdot 1.05^{10} \approx \$814.45$.
11. **Q.** Half-life decay: 80 g halves every hour. Mass after 5 hours?
    A. $80 \cdot (1/2)^5 = 80 \cdot 1/32 = 2.5$ g.
12. **Q.** A geometric sequence has $a_1 = 4$, $a_5 = 324$. Find r.
    A. $324 = 4r^4 \rightarrow r^4 = 81 \rightarrow r = 3$.
13. **Q.** Is 1, 4, 9, 16 arithmetic, geometric, or neither?
    A. Neither — these are perfect squares.
14. **Q.** Sum $1 + 2 + 4 + 8 + \ldots + 1024$.
    A. $S = 1(1 - 2^{11})/(1 - 2) = 2047$.
15. **Q.** A car worth $20,000 depreciates 15% per year. Value after 5 years?
    A. $20000 \cdot 0.85^5 \approx \$8,874$.

Quiz Answers

1. **Answer: (C) Multiplying by r**
   Reason: Constant ratio r distinguishes geometric.
2. **Answer: (C) 2**
   Reason: 8/4 = 2.

3. **Answer: (C) 768**

   Reason: $3 \cdot 4^4 = 3 \cdot 256 = 768.$

4. **Answer: (B) Arithmetic**

   Reason: *Constant difference 2.*

5. **Answer: (D) Geometric**

   Reason: *Constant ratio 2.*

6. **Answer: (A) 31**

   Reason: $S_5 = 1(1-2^5)/(1-2) = 31.$

7. **Answer: (C) 1600**

   Reason: $100 \cdot 2^4 = 1600.$

8. **Answer: (B) ≈ $1791**

   Reason: $1000 \cdot 1.06^{10} \approx \$1790.85.$

ANSWER KEY

Week 30 — Summation Formulas and Applications

Practice Exercises

1. **Q.** Evaluate $\sum_{i=1}^{6} i$.
   A. 6·7/2 = 21.
2. **Q.** Evaluate $\sum_{i=1}^{10} i$.
   A. 10·11/2 = 55.
3. **Q.** Evaluate $\sum_{i=1}^{5} 4$.
   A. 5 · 4 = 20.
4. **Q.** Sum 3+6+9+12+15.
   A. S = 5/2 · 18 = 45.
5. **Q.** Geometric sum 5+10+20+40.
   A. S_4 = 5(1−16)/(1−2) = 75.
6. **Q.** $\sum_{i=1}^{3} (i + 2)$ = ?
   A. Split: (1+2+3) + 3·2 = 6 + 6 = 12.
7. **Q.** $\sum_{i=1}^{4} (2i - 1)$ = ?
   A. 1+3+5+7 = 16.
8. **Q.** Sum of infinite series 1 + 0.1 + 0.01 + 0.001 + …
   A. S_∞ = 1/(1−0.1) = 10/9.
9. **Q.** Sum of infinite series 4 + 2 + 1 + 0.5 + …
   A. S_∞ = 4/(1−0.5) = 8.
10. **Q.** Save $20 first week, +$5 each week. Total after 8 weeks?
    A. a_1=20, a_8=55. S = 8/2 · 75 = 300. $300.
11. **Q.** A theatre has 30 seats in row 1, +3 per row. Total in 12 rows?
    A. a_1=30, a_{12}=63. S = 12/2 · 93 = 558.
12. **Q.** Bouncing ball: drop from 10 m, bounces 60% each time. Total distance?
    A. 10 + 2·(10·0.6)/(1−0.6) = 10 + 30 = 40 m.
13. **Q.** Express 1+2+3+…+100 in sigma notation.
    A. $\sum_{i=1}^{100} i$.
14. **Q.** Express 2+4+6+…+100 in sigma notation.
    A. $\sum_{i=1}^{50} 2i$.
15. **Q.** Repeating decimal 0.999… as an infinite geometric series equals…
    A. 0.9 + 0.09 + 0.009 + … = 0.9/(1−0.1) = 1.

Quiz Answers

1. **Answer: (D) 10**
   Reason: 1 + 2 + 3 + 4 = 10.
2. **Answer: (B) 21**
   Reason: 6·7/2 = 21.

3. **Answer: (B) 15**

   Reason: 5 copies of 3 = 15.

4. **Answer: (A) 25**

   Reason: $a_1=1$, $a_5=9$, $n=5$. $S = 5/2 \cdot 10 = 25$.

5. **Answer: (D) 40**

   Reason: $S_4 = 1(1-81)/(1-3) = 40$.

6. **Answer: (C) 15**

   Reason: Split: $2(1+2+3) + 3 = 12 + 3 = 15$.

7. **Answer: (B) 2**

   Reason: $S_\infty = 1/(1-1/2) = 2$.

8. **Answer: (A) $185**

   Reason: $a_1=5$, $a_{10}=32$. $S = 10/2 \cdot 37 = 185$.

UNIT 8
Probability and Statistics

UNIT OVERVIEW

How likely is something? Counting techniques, basic probability rules, descriptive statistics (mean, median, standard deviation), and an introduction to distributions — the mathematics behind data science, polling, insurance, and games of chance.

Weeks in this unit:

Week 31 — *Combinatorics — Permutations and Combinations*
Week 32 — *Probability and Data Interpretation*

UNIT 8 · PROBABILITY AND STATISTICS

WEEK 31

Combinatorics — Permutations and Combinations

The mathematics of counting without listing. Master factorials, permutations (order matters), and combinations (order does not) — the foundations of probability, gambling odds, and lottery math.

Learning Objectives

By the end of this week, you will be able to:

1. Define permutations and combinations.
2. Understand factorial notation.
3. Apply permutation formulas correctly.
4. Apply combination formulas correctly.
5. Distinguish between order-sensitive and order-insensitive problems.
6. Solve basic counting problems.
7. Apply combinatorics to real-life situations.
8. Choose appropriate counting methods for problems.

Key Concepts

1. Factorials

The factorial of n, written n exclamation mark, is the product of every positive integer from one up to n. Factorials grow extremely fast.

2. Permutations — order matters

A permutation is an arrangement where order matters. Choosing 1st, 2nd, and 3rd place from a group is a permutation.

3. Combinations — order does not matter

A combination is a selection where order does not matter. Choosing a committee of three from a group is a combination.

4. Permutation or combination?

If the order of selection changes the result, it is a permutation. If not, it is a combination.

5. Worked examples

Five worked counting examples.

Worked Examples

Example 1. Compute 4!

Solution: 24.

Example 2. Compute 7!

Solution: 5040.

Example 3. $P(5, 2) = ?$

Solution: $5 \cdot 4 = 20$.

UNIT 8 · PROBABILITY AND STATISTICS

Week 31 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Compute 4!

2. Compute 7!

3. P(5, 2) = ?

4. P(6, 3) = ?

5. C(6, 2) = ?

6. C(8, 3) = ?

7. C(10, 4) = ?

8. Team captain and vice-captain from 12 players: permutation or combination?

9. Pick 3 toppings from 10: how many ways?

10. Select 4 students from 10 for a committee.

11. How many 4-digit passwords (digits can repeat) from 0-9?

12. Arrangements of letters in MATH?

13. Arrange 5 books on a shelf.

14. Choose 5 cards from 52 for a poker hand.

15. Pick 6 numbers from 49 for a lottery.

UNIT 8 · PROBABILITY AND STATISTICS

Week 31 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. 5! = ?

(A) 15
(B) 60
(C) 120
(D) 25
Your answer: ____________

2. 7! = ?

(A) 720
(B) 5040
(C) 49
(D) 42
Your answer: ____________

3. P(5, 2) = ?

(A) 120
(B) 10
(C) 7
(D) 20
Your answer: ____________

4. C(6, 2) = ?

(A) 15
(B) 3
(C) 12
(D) 30
Your answer: ____________

5. Awarding gold, silver, bronze from 8 runners is…

(A) Combination
(B) Neither
(C) Both
(D) Permutation
Your answer: ____________

6. Choosing 5 cards from a deck for a hand is…

(A) Permutation
(B) Neither
(C) Combination

(D) Both

Your answer: ____________

7. How many ways to arrange letters in WATER?

(A) 5
(B) 24
(C) 120
(D) 25

Your answer: ____________

8. A class of 10 selects 3 students for a committee. How many ways?

(A) 30
(B) 10
(C) 120
(D) 720

Your answer: ____________

DID YOU KNOW?

A standard deck of 52 cards can be shuffled into 52! different orders — about 8 × 10^67. That number is so large that, if every atom in the visible universe were a separate shuffled deck, we would still have shuffled fewer than one in a billion possible orders. Every time you shuffle a deck well, you almost certainly create an arrangement that has never existed before in history.

UNIT 8 · PROBABILITY AND STATISTICS

WEEK 32

Probability and Data Interpretation

How likely is something? Master the basic probability formula, sample spaces and events, the difference between theoretical and experimental probability, and the three measures of central tendency.

Learning Objectives

By the end of this week, you will be able to:

1. Define probability and key terms.
2. Distinguish between theoretical and experimental probability.
3. Calculate basic probabilities using formulas.
4. Identify sample spaces and events.
5. Interpret data from graphs and tables.
6. Calculate and interpret mean, median, and mode.
7. Apply probability to real-life situations.
8. Draw conclusions from data representations.

Key Concepts

1. What is probability?

Probability measures how likely an event is to happen. It is a number between zero (impossible) and one (certain).

2. Sample space and events

The sample space is the set of all possible outcomes. An event is a subset of the sample space.

3. Theoretical vs experimental

Theoretical probability is based on math. Experimental probability is based on what actually happened over many trials.

4. Measures of central tendency

Three ways to summarize a data set: the mean (average), the median (middle value), and the mode (most frequent value).

5. Worked examples

Five worked probability and statistics examples.

Worked Examples

Example 1. P(rolling 3 on a die)?

Solution: 1/6.

Example 2. P(heads on a coin)?

Solution: 1/2.

Example 3. List sample space for two coin tosses.

Solution: {HH, HT, TH, TT}.

UNIT 8 · PROBABILITY AND STATISTICS

Week 32 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. P(rolling 3 on a die)?

2. P(heads on a coin)?

3. List sample space for two coin tosses.

4. List sample space for rolling two dice (size).

5. Mean of 5, 10, 15, 20.

6. Median of 4, 8, 12, 16, 20.

7. Mode of 1, 2, 2, 3, 4.

8. Bag has 3 red, 2 blue, 5 green. P(red)?

9. Same bag, P(blue)?

10. P(not green) for the same bag.

11. P(rolling a number greater than 4)?

12. A spinner has 4 equal regions: red, blue, green, yellow. P(red or blue)?

13. In 200 coin flips you get 110 heads. Experimental P(heads)?

14. Theoretical P(heads) for a fair coin?

15. Find mean, median, mode of 4, 4, 5, 7, 10.

UNIT 8 · PROBABILITY AND STATISTICS

Week 32 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. P(rolling a 5 on a fair die)?

(A) 1/6
(B) 5
(C) 5/6
(D) 1/2
Your answer: ____________

2. P(heads on a fair coin)?

(A) 0
(B) 1
(C) 1/2
(D) 1/4
Your answer: ____________

3. Sample space for tossing 2 coins has how many outcomes?

(A) 2
(B) 4
(C) 6
(D) 8
Your answer: ____________

4. Mean of 2, 4, 6, 8?

(A) 7
(B) 6
(C) 5
(D) 4
Your answer: ____________

5. Median of 3, 5, 7, 9, 11?

(A) 9
(B) 11
(C) 5
(D) 7
Your answer: ____________

6. Mode of 2, 3, 3, 5, 7?

(A) 5
(B) 3
(C) 2

(D) 7

Your answer: ____________

7. Bag has 3 red, 2 blue, 5 green. P(blue)?

(A) 2/10

(B) 3/10

(C) 1/2

(D) 5/10

Your answer: ____________

8. A coin is flipped 100 times yielding 47 heads. Experimental P(heads) is…

(A) 0.5

(B) 1

(C) 0.53

(D) 0.47

Your answer: ____________

DID YOU KNOW?

Modern probability theory was born from a gambling problem. In 1654, the French nobleman Chevalier de Méré asked Blaise Pascal how to fairly divide stakes if a dice game ended early. Pascal wrote to Fermat, the two exchanged letters, and from that correspondence emerged the entire mathematical framework now used by insurance companies, weather forecasters, quantum physicists, and machine learning engineers. A gambling question started a science.

ANSWER KEY

Unit 8 · Probability and Statistics

This answer key covers every practice exercise and quiz question from Unit 8. For full step-by-step solutions to randomised practice generators (separate from the worksheet exercises printed here), refer to the BemandaSTEM Precalculus app.

ANSWER KEY

Week 31 — Combinatorics — Permutations and Combinations

Practice Exercises

1. **Q.** Compute 4!
   A. 24.
2. **Q.** Compute 7!
   A. 5040.
3. **Q.** P(5, 2) = ?
   A. 5·4 = 20.
4. **Q.** P(6, 3) = ?
   A. 6·5·4 = 120.
5. **Q.** C(6, 2) = ?
   A. 15.
6. **Q.** C(8, 3) = ?
   A. 56.
7. **Q.** C(10, 4) = ?
   A. 210.
8. **Q.** Team captain and vice-captain from 12 players: permutation or combination?
   A. Permutation. P(12, 2) = 132.
9. **Q.** Pick 3 toppings from 10: how many ways?
   A. C(10, 3) = 120.
10. **Q.** Select 4 students from 10 for a committee.
    A. C(10, 4) = 210.
11. **Q.** How many 4-digit passwords (digits can repeat) from 0-9?
    A. 10^4 = 10,000.
12. **Q.** Arrangements of letters in MATH?
    A. 4! = 24.
13. **Q.** Arrange 5 books on a shelf.
    A. 5! = 120.
14. **Q.** Choose 5 cards from 52 for a poker hand.
    A. C(52, 5) = 2,598,960.
15. **Q.** Pick 6 numbers from 49 for a lottery.
    A. C(49, 6) = 13,983,816.

Quiz Answers

1. **Answer: (C) 120**
   Reason: 5·4·3·2·1 = 120.
2. **Answer: (B) 5040**
   Reason: 7·6·5·4·3·2·1 = 5040.

3. **Answer: (D) 20**

   Reason: $5!/3! = 5 \cdot 4 = 20$.

4. **Answer: (A) 15**

   Reason: $6!/(2! \cdot 4!) = 15$.

5. **Answer: (D) Permutation**

   Reason: Order matters — different medals → permutation.

6. **Answer: (C) Combination**

   Reason: Order does not matter in a hand of cards.

7. **Answer: (C) 120**

   Reason: $5! = 120$.

8. **Answer: (C) 120**

   Reason: $C(10, 3) = 120$.

ANSWER KEY

Week 32 — Probability and Data Interpretation

Practice Exercises

1. **Q.** P(rolling 3 on a die)?
   A. 1/6.
2. **Q.** P(heads on a coin)?
   A. 1/2.
3. **Q.** List sample space for two coin tosses.
   A. {HH, HT, TH, TT}.
4. **Q.** List sample space for rolling two dice (size).
   A. 36 outcomes.
5. **Q.** Mean of 5, 10, 15, 20.
   A. 50/4 = 12.5.
6. **Q.** Median of 4, 8, 12, 16, 20.
   A. 12.
7. **Q.** Mode of 1, 2, 2, 3, 4.
   A. 2.
8. **Q.** Bag has 3 red, 2 blue, 5 green. P(red)?
   A. 3/10.
9. **Q.** Same bag, P(blue)?
   A. 2/10 = 1/5.
10. **Q.** P(not green) for the same bag.
    A. 5/10 = 1/2.
11. **Q.** P(rolling a number greater than 4)?
    A. {5, 6}: 2/6 = 1/3.
12. **Q.** A spinner has 4 equal regions: red, blue, green, yellow. P(red or blue)?
    A. 2/4 = 1/2.
13. **Q.** In 200 coin flips you get 110 heads. Experimental P(heads)?
    A. 110/200 = 0.55.
14. **Q.** Theoretical P(heads) for a fair coin?
    A. 0.5.
15. **Q.** Find mean, median, mode of 4, 4, 5, 7, 10.
    A. Mean = 6, Median = 5, Mode = 4.

Quiz Answers

1. **Answer: (A) 1/6**
   Reason: One favourable out of 6.
2. **Answer: (C) 1/2**
   Reason: 1 favourable out of 2 → 1/2.

3. **Answer: (B) 4**

   Reason: {HH, HT, TH, TT} → 4.

4. **Answer: (C) 5**

   Reason: (2+4+6+8)/4 = 5.

5. **Answer: (D) 7**

   Reason: Middle value.

6. **Answer: (B) 3**

   Reason: 3 appears twice — most frequent.

7. **Answer: (A) 2/10**

   Reason: 2 blue out of 10 total = 2/10 = 1/5.

8. **Answer: (D) 0.47**

   Reason: 47/100 = 0.47.

UNIT 9
Introduction to Limits

UNIT OVERVIEW

The first taste of calculus. Discover what happens to a function as the input approaches a value — including infinity. Master limit notation, evaluation rules, continuity, and the limit definitions that lead directly into derivatives next year.

Weeks in this unit:

Week 33 — *Concept of a Limit*
Week 34 — *Function Behavior Near a Point*
Week 35 — *Limits of Algebraic Functions*
Week 36 — *Applications & Final Review*

UNIT 9 · INTRODUCTION TO LIMITS

WEEK 33

Concept of a Limit

Welcome to the first idea of calculus. A limit describes what a function is approaching as x approaches a value — even if the function never actually reaches it.

Learning Objectives

By the end of this week, you will be able to:

1. Define the concept of a limit.
2. Interpret limit notation correctly.
3. Evaluate simple limits numerically and graphically.
4. Distinguish between left-hand and right-hand limits.
5. Apply basic limit properties.
6. Understand limits of polynomial functions.
7. Relate limits to real-world situations.
8. Recognise the connection between limits and continuity.

Key Concepts

1. What is a limit?

A limit describes the value a function approaches as its input gets closer to a target. The limit asks: where is the function heading, not where it lands.

2. Left-hand and right-hand limits

A function can be approached from the left or from the right. The full limit exists only if both one-sided limits agree.

3. How to evaluate limits

Start with direct substitution: plug a into f(x). If you get a real number, that is the answer.

4. Limit properties

Limits obey simple algebra rules. The limit of a sum is the sum of limits. The limit of a product is the product of limits. Constants pass through.

5. Worked examples

Five worked limit examples.

Worked Examples

Example 1. $\lim_{x \to 4}(x + 2) = ?$

Solution: 6.

Example 2. $\lim_{x \to 1}(x^2 + 1) = ?$

Solution: 2.

Example 3. $\lim_{x \to 0}(3x^2 - 2x + 1) = ?$

Solution: 1.

UNIT 9 · INTRODUCTION TO LIMITS

Week 33 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. $\lim_{x \to 4}(x + 2) = ?$

2. $\lim_{x \to 1}(x^2 + 1) = ?$

3. $\lim_{x \to 0}(3x^2 - 2x + 1) = ?$

4. $\lim_{x \to 2}(x^2 + 3x) = ?$

5. $\lim_{x \to 5} 7x = ?$

6. $\lim_{x \to 3}(x + 4)/(x - 1) = ?$

7. If left limit = 4 and right limit = 4, does limit exist?

8. If left = 3 and right = 7, does limit exist?

9. Use sum rule on $\lim_{x \to 2}(x + x^2)$.

10. Use product rule on $\lim_{x \to 3} (x)(x + 1)$.

11. $\lim_{x \to 0} (x^2 + 5)/(x + 2) = ?$

12. What does $\lim_{x \to \infty} 1/x$ equal?

13. What does $\lim_{x \to 0^+} 1/x$ equal?

14. $\lim_{x \to 2} 3(x^2) = ?$

15. Is the limit of a polynomial at a real number always equal to its value there?

UNIT 9 · INTRODUCTION TO LIMITS

Week 33 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. $\lim_{x\to 4}(x + 2) = ?$

- **(A)** 4
- **(B)** 6
- **(C)** 2
- **(D)** 8

Your answer: ____________

2. $\lim_{x\to 1} (x^2 + 1) = ?$

- **(A)** 3
- **(B)** 0
- **(C)** 2
- **(D)** 1

Your answer: ____________

3. $\lim_{x\to 3} x^3 = ?$

- **(A)** 27
- **(B)** 12
- **(C)** 9
- **(D)** 6

Your answer: ____________

4. If left-hand = 5 and right-hand = 5, the limit equals…

- **(A)** Does not exist
- **(B)** 10
- **(C)** 0
- **(D)** 5

Your answer: ____________

5. If left-hand = 2 and right-hand = 7, the limit…

- **(A)** 4.5
- **(B)** 9
- **(C)** 5
- **(D)** Does not exist

Your answer: ____________

6. $\lim_{x\to 2} 5x = ?$

- **(A)** 25
- **(B)** 2
- **(C)** 7

(D) 10

Your answer: ____________

7. $\lim_{x \to 0} (x^2 + 3x + 1) = ?$

(A) 3
(B) 4
(C) 1
(D) 0

Your answer: ____________

8. A limit is about...

(A) The derivative
(B) Where the function is approaching
(C) The slope
(D) Where the function lands

Your answer: ____________

DID YOU KNOW?

Isaac Newton and Gottfried Leibniz independently invented calculus in the 1660s and 1670s, but neither could rigorously explain what a limit was. They used vague notions of "vanishing quantities" — and it worked! Calculus was used to predict planet orbits and design machines for almost two centuries before Augustin-Louis Cauchy and Karl Weierstrass finally gave limits a rigorous epsilon-delta definition in the 1820s and 1850s. The single most useful idea in mathematics was used productively for 200 years before anyone could properly say what it meant.

UNIT 9 · INTRODUCTION TO LIMITS

WEEK 34

Function Behavior Near a Point

Functions can do strange things at specific points — holes, jumps, vertical blow-ups. Learn to recognize and classify the four kinds of local behavior that limits help us describe.

Learning Objectives

By the end of this week, you will be able to:

1. Describe how functions behave near a given point.
2. Analyse left-hand and right-hand behaviour of functions.
3. Use tables and graphs to estimate function values near a point.
4. Identify continuity and discontinuity informally.
5. Recognise different types of discontinuities.
6. Interpret real-world situations using function behaviour.
7. Estimate function values when direct substitution fails.
8. Connect function behaviour to the idea of limits.

Key Concepts

1. Looking near a point

Function behavior near a point is the study of what a function does as the input approaches a specific value. We care about nearby values, not necessarily the value at the point itself.

2. The four kinds of local behavior

Smooth, hole, jump, infinite. Four kinds of behavior a function can have near a point.

3. Holes and removable discontinuities

A removable discontinuity happens when a factor cancels in a rational function, leaving a single missing point in an otherwise smooth graph.

4. Jumps and vertical asymptotes

A jump discontinuity has different left and right values. A vertical asymptote sends the function shooting toward infinity.

5. Worked examples

Five worked examples of local function behavior.

Worked Examples

Example 1. Classify $f(x) = x^3$ at $x = 2$.

Solution: Smooth — polynomials are continuous everywhere.

Example 2. Classify $f(x) = 1/(x - 1)$ at $x = 1$.

Solution: Vertical asymptote.

Example 3. Classify $f(x) = (x^2 - 16)/(x - 4)$ at $x = 4$.

Solution: Hole (removable). Simplifies to $x + 4$, value 8.

UNIT 9 · INTRODUCTION TO LIMITS

Week 34 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Classify $f(x) = x^3$ at $x = 2$.

2. Classify $f(x) = 1/(x - 1)$ at $x = 1$.

3. Classify $f(x) = (x^2 - 16)/(x - 4)$ at $x = 4$.

4. Find the value at the hole of $(x^2 - 25)/(x - 5)$.

5. Find $\lim_{x \to 2^+} 1/(x - 2)$.

6. Find $\lim_{x \to 2^-} 1/(x - 2)$.

7. A function has left = 3 and right = 7 at x = 0. Does limit exist?

8. Is $f(x) = \sqrt{x}$ continuous at $x = 0$?

9. Find limit of $f(x) = (x - 1)/(x - 1)$ as $x \to 1$.

10. Identify type: a sensor jumps from 5 V to 15 V at $t = 0$.

11. Identify type: a smooth temperature curve rising through 20°C.

12. Identify type: division by zero producing infinite voltage.

13. A table gives $f(2.9) = 5.81$, $f(2.99) = 5.98$, $f(3.01) = 6.02$, $f(3.1) = 6.21$. Estimate the limit.

14. Why does $(x - 2)/(x - 2)$ have a hole at $x = 2$ even though it simplifies to 1?

15. What is the limit of $1/x$ as x approaches infinity?

UNIT 9 · INTRODUCTION TO LIMITS

Week 34 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. A function with a missing point but consistent surrounding values has a…

(A) No discontinuity
(B) Vertical asymptote
(C) Jump
(D) Removable discontinuity (hole)
Your answer: ____________

2. A piecewise function with different left and right values at $x = a$ has a…

(A) Jump discontinuity
(B) Smooth point
(C) Vertical asymptote
(D) Hole
Your answer: ____________

3. $f(x) = 1/(x - 4)$ near $x = 4$ has a…

(A) Smooth point
(B) Vertical asymptote
(C) Hole
(D) Jump
Your answer: ____________

4. $f(x) = (x^2 - 9)/(x - 3)$ at $x = 3$ has a…

(A) Jump
(B) Asymptote
(C) No discontinuity
(D) Hole
Your answer: ____________

5. Find the limit of $(x^2 - 4)/(x - 2)$ as $x \to 2$.

(A) DNE
(B) 2
(C) 0
(D) 4
Your answer: ____________

6. Polynomials are continuous…

(A) Everywhere
(B) Never
(C) Only at zero

(D) Only at integers

Your answer: ____________

7. If left = 4 and right = 4, the limit equals...

(A) 8
(B) DNE
(C) 0
(D) 4

Your answer: ____________

8. $\lim_{x \to 3^+} 1/(x - 3) = ?$

(A) 1
(B) $-\infty$
(C) $+\infty$
(D) 0

Your answer: ____________

DID YOU KNOW?

The function f(x) = sin(x)/x has a removable hole at x = 0, but the limit is exactly 1 — an extraordinary fact discovered by mathematicians in the 1600s. This single limit underlies the entire foundation of calculus for trigonometric functions. Without it, we would not have the derivative of sine, and without the derivative of sine, modern physics, engineering, and signal processing would not exist.

UNIT 9 · INTRODUCTION TO LIMITS

WEEK 35

Limits of Algebraic Functions

When direct substitution gives 0/0, you need to do algebra first. Factor, cancel, then substitute. Master the techniques that handle the indeterminate forms calculus throws at you.

Learning Objectives

By the end of this week, you will be able to:

1. Evaluate limits of algebraic functions using substitution.
2. Simplify expressions using factoring before evaluating limits.
3. Solve limits that produce indeterminate forms.
4. Evaluate limits of rational functions.
5. Apply algebraic techniques to resolve 0/0 forms.
6. Interpret algebraic limits graphically and numerically.
7. Solve real-world problems using limits.
8. Strengthen understanding of limit laws in algebraic contexts.

Key Concepts

1. Direct substitution first

Always try plugging in the value first. For polynomials and most algebraic expressions, direct substitution gives the answer immediately.

2. The 0/0 indeterminate form

When substitution gives zero over zero, the limit is not necessarily undefined. It could equal anything — you must simplify first.

3. The factoring strategy

When the numerator and denominator both equal zero at x equals a, both must contain x minus a as a factor. Factor it out, cancel, and substitute.

4. Limits with square roots

For limits involving square roots, multiply numerator and denominator by the conjugate to rationalize and reveal the cancelling factor.

5. Worked examples

Five worked algebraic limit examples.

Worked Examples

Example 1. $\lim_{x \to 1} (x^2 + 2x) = ?$

Solution: $1 + 2 = 3$.

Example 2. $\lim_{x \to 0} (3x^2 + 5x - 1) = ?$

Solution: -1.

Example 3. $\lim_{x \to 3} (x^2 - 9)/(x - 3) = ?$

Solution: Factor: $x + 3 \to 6$.

UNIT 9 · INTRODUCTION TO LIMITS

Week 35 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. $\lim_{x \to 1} (x^2 + 2x) = ?$

2. $\lim_{x \to 0} (3x^2 + 5x - 1) = ?$

3. $\lim_{x \to 3} (x^2 - 9)/(x - 3) = ?$

4. $\lim_{x \to 4} (x^2 - 16)/(x - 4) = ?$

5. $\lim_{x \to 2} (x^3 - 8)/(x - 2) = ?$

6. $\lim_{x \to -1} (x^2 + 2x + 1)/(x + 1) = ?$

7. $\lim_{x \to 5} (x^2 - 5x)/(x - 5) = ?$

8. $\lim_{x \to 4} (\sqrt{x} - 2)/(x - 4) = ?$

9. $\lim_{x \to 0} (\sqrt{(x+1)} - 1)/x = ?$

10. Why is direct substitution always tried first?

11. $\lim_{x \to 2} (x^2 - 3x + 2)/(x - 2) = ?$

12. $\lim_{x \to 3} (x^2 - 4x + 3)/(x - 3) = ?$

13. If both numerator and denominator → 0 at x = a, what does that tell you?

14. $\lim_{x \to 5} (x - 5)/(x^2 - 25) = ?$

15. $\lim_{x \to -2} (x^2 - 4)/(x + 2) = ?$

UNIT 9 · INTRODUCTION TO LIMITS

Week 35 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. $\lim_{x \to 1} (x^2 + 2x) = ?$

- **(A)** 0
- **(B)** 1
- **(C)** 2
- **(D)** 3

Your answer: ____________

2. $\lim_{x \to 3} (x^2 - 9)/(x - 3) = ?$

- **(A)** 6
- **(B)** 9
- **(C)** 3
- **(D)** 0

Your answer: ____________

3. $\lim_{x \to 4} (x^2 - 16)/(x - 4) = ?$

- **(A)** 8
- **(B)** 16
- **(C)** 4
- **(D)** 0

Your answer: ____________

4. $\lim_{x \to 2} (x^2 - x - 2)/(x - 2) = ?$

- **(A)** 3
- **(B)** 0
- **(C)** 1
- **(D)** 2

Your answer: ____________

5. What does 0/0 mean in a limit?

- **(A)** Always equals 1
- **(B)** Always undefined
- **(C)** Always equals 0
- **(D)** Indeterminate — simplify and re-evaluate

Your answer: ____________

6. Best method for $\lim_{x \to 1} (3x + 5)$?

- **(A)** Direct substitution
- **(B)** Factor
- **(C)** Derivative rule

(D) Rationalise
Your answer: ____________

7. Best method for $\lim_{x \to 4} (\sqrt{x} - 2)/(x - 4)$?

(A) Factor only
(B) Direct substitution
(C) Rationalise (conjugate)
(D) Just guess
Your answer: ____________

8. $\lim_{x \to 5} (x^2 - 25)/(x - 5) = ?$

(A) 0
(B) 10
(C) 25
(D) 5
Your answer: ____________

DID YOU KNOW?

The expression 0/0 was called an "indeterminate form" by 19th-century mathematicians because — unlike 1/0 or ∞/∞ — it can equal any real number depending on how you got there. The limit $(x^2 - 4)/(x - 2)$ equals 4 at $x = 2$, but the limit $\sin(x)/x$ equals 1 at $x = 0$, and $(1 - \cos x)/x$ equals 0 at $x = 0$. All three look like 0/0, but each gives a different finite answer. This is why we cannot just "evaluate" 0/0 — we have to factor or rationalize.

UNIT 9 · INTRODUCTION TO LIMITS

WEEK 36

Applications & Final Review

The capstone. Synthesise every tool from 36 weeks of precalculus — functions, polynomials, exponentials, trigonometry, conics, systems, sequences, probability, and limits. Mathematics is one connected story.

Learning Objectives

By the end of this week, you will be able to:

1. Recall and apply major concepts from the full course.
2. Solve mixed problems involving multiple topics.
3. Select appropriate methods for different problem types.
4. Interpret graphs, equations, and data sets.
5. Connect algebra, geometry, and calculus concepts.
6. Solve real-world multi-step problems.
7. Demonstrate fluency in mathematical reasoning.
8. Prepare effectively for final assessments.

Key Concepts

1. The whole story

Across nine units and thirty-six weeks, you have built a complete picture of precalculus mathematics — from basic functions to the first glimpses of calculus.

2. The five great formula families

Five formula families cover most of precalculus. Memorise these and you can solve almost anything.

3. How math topics connect

Functions describe relationships. Geometry gives them shape. Trig adds rotation. Sequences add steps. Limits add motion.

4. Looking forward to calculus

Calculus has two big ideas: derivatives (instantaneous rate of change) and integrals (accumulated total). Both are built on limits.

5. Worked synthesis examples

Five capstone examples drawing on multiple units.

Worked Examples

Example 1. Distance between (1, 2) and (5, 6)?

Solution: $\sqrt{(16 + 16)} = \sqrt{32} \approx 5.66$.

Example 2. Midpoint of (3, 7) and (9, 1)?

Solution: (6, 4).

Example 3. Slope between (2, 3) and (5, 9)?

Solution: $(9 - 3)/(5 - 2) = 2$.

UNIT 9 · INTRODUCTION TO LIMITS

Week 36 — Practice Exercises

Complete each exercise in the space provided. Show all working. Check your answers using the answer key at the end of this unit.

1. Distance between (1, 2) and (5, 6)?

2. Midpoint of (3, 7) and (9, 1)?

3. Slope between (2, 3) and (5, 9)?

4. Solve $x^2 - 5x + 6 = 0$.

5. Evaluate $\sin(\pi/3)$.

6. Evaluate $\cos(\pi/4)$.

7. $\log_2(16) = ?$

8. $2^3 \times 2^4 = ?$

9. 10th term of $a_1 = 2$, $d = 4$?

10. Sum of first 10 terms of $a_1 = 1$, $d = 2$?

11. Identify $x^2 + y^2 = 25$.

12. Identify $y = x^2 - 4x + 3$.

13. $\lim_{x \to 3} (x + 2) = ?$

14. $\lim_{x \to 4} (x^2 - 16)/(x - 4) = ?$

15. A bag has 4 red, 3 blue, 3 green. Find P(not red).

16. How many ways to arrange the letters in MATH?

17. Find sum 1+2+3+...+100.

18. A salary starts at $40,000 with annual raises of $2,000. Salary in year 10?

19. Total earnings over 10 years (sequence problem).

20. A right triangle has angles 30°, 60°, 90° and hypotenuse 10. Find the shorter leg.

UNIT 9 · INTRODUCTION TO LIMITS

Week 36 — Quiz Practice

Choose the best answer for each question. Circle your choice or write the letter (A, B, C, or D) in the box provided. A score of 80% (7 of 8 correct) shows mastery of this week's material.

1. Distance between (0, 0) and (3, 4)?

(A) $\sqrt{7}$
(B) 12
(C) 5
(D) 7
Your answer: ____________

2. 10th term of an arithmetic sequence with $a_1 = 5$, $d = 3$?

(A) 35
(B) 32
(C) 15
(D) 30
Your answer: ____________

3. Hypotenuse of right triangle with legs 5 and 12?

(A) 7
(B) 60
(C) 17
(D) 13
Your answer: ____________

4. P(rolling 3 or more on a die)?

(A) 3/6
(B) 4/6
(C) 2/6
(D) 5/6
Your answer: ____________

5. $\lim_{x \to 5} (x^2 - 25)/(x - 5) = ?$

(A) 5
(B) 0
(C) 25
(D) 10
Your answer: ____________

6. $f(x) = 3^x$ is what type of function?

(A) Polynomial
(B) Linear
(C) Exponential

(D) Logarithmic

Your answer: ____________

7. $x^2/16 + y^2/9 = 1$ represents a…

(A) Hyperbola
(B) Ellipse
(C) Parabola
(D) Circle

Your answer: ____________

8. Sum of first 100 positive integers?

(A) 1000
(B) 10100
(C) 100
(D) 5050

Your answer: ____________

DID YOU KNOW?

Congratulations — you have completed Grade 12 Precalculus. Across these 36 weeks, you have studied the same mathematics that took the human race over 2,500 years to develop. From Euclid in 300 BCE, to al-Khwarizmi inventing algebra in 820 CE, to Descartes uniting algebra and geometry in 1637, to Newton and Leibniz inventing calculus in the 1670s, to Cauchy and Weierstrass rigorising limits in the 1800s — every mathematical idea you mastered this year was a hard-won discovery by brilliant minds across cultures and centuries. You now stand on their shoulders. Next year, calculus awaits.

ANSWER KEY

Unit 9 · Introduction to Limits

This answer key covers every practice exercise and quiz question from Unit 9. For full step-by-step solutions to randomised practice generators (separate from the worksheet exercises printed here), refer to the BemandaSTEM Precalculus app.

ANSWER KEY

Week 33 — Concept of a Limit

Practice Exercises

1. **Q.** $\lim_{x\to 4}(x + 2) = ?$

   A. 6.

2. **Q.** $\lim_{x\to 1}(x^2 + 1) = ?$

   A. 2.

3. **Q.** $\lim_{x\to 0}(3x^2 - 2x + 1) = ?$

   A. 1.

4. **Q.** $\lim_{x\to 2}(x^2 + 3x) = ?$

   A. $4 + 6 = 10$.

5. **Q.** $\lim_{x\to 5} 7x = ?$

   A. 35.

6. **Q.** $\lim_{x\to 3}(x + 4)/(x - 1) = ?$

   A. $7/2 = 3.5$.

7. **Q.** If left limit = 4 and right limit = 4, does limit exist?

   A. Yes, equals 4.

8. **Q.** If left = 3 and right = 7, does limit exist?

   A. No, DNE.

9. **Q.** Use sum rule on $\lim_{x\to 2}(x + x^2)$.

   A. $\lim x + \lim x^2 = 2 + 4 = 6$.

10. **Q.** Use product rule on $\lim_{x\to 3} (x)(x + 1)$.

    A. $3 \cdot 4 = 12$.

11. **Q.** $\lim_{x\to 0} (x^2 + 5)/(x + 2) = ?$

    A. $5/2 = 2.5$.

12. **Q.** What does $\lim_{x\to \infty} 1/x$ equal?

    A. 0 (as x grows, 1/x shrinks to 0).

13. **Q.** What does $\lim_{x\to 0^+} 1/x$ equal?

    A. $+\infty$ (grows without bound from the right).

14. **Q.** $\lim_{x\to 2} 3(x^2) = ?$

    A. $3 \cdot 4 = 12$.

15. **Q.** Is the limit of a polynomial at a real number always equal to its value there?

    A. Yes — polynomials are continuous everywhere, so substitute directly.

Quiz Answers

1. **Answer: (B) 6**

   Reason: Direct sub: $4 + 2 = 6$.

2. **Answer: (C) 2**

   Reason: $1^2 + 1 = 2$.

3. **Answer: (A) 27**

   Reason: $3^3 = 27$.

4. **Answer: (D) 5**

   Reason: Both sides agree → limit = 5.

5. **Answer: (D) Does not exist**

   Reason: Sides disagree → DNE.

6. **Answer: (D) 10**

   Reason: $5 \cdot 2 = 10$.

7. **Answer: (C) 1**

   Reason: $0 + 0 + 1 = 1$.

8. **Answer: (B) Where the function is approaching**

   Reason: Limits describe approach — not necessarily the actual function value.

ANSWER KEY

Week 34 — Function Behavior Near a Point

Practice Exercises

1. **Q.** Classify $f(x) = x^3$ at $x = 2$.
   A. Smooth — polynomials are continuous everywhere.
2. **Q.** Classify $f(x) = 1/(x - 1)$ at $x = 1$.
   A. Vertical asymptote.
3. **Q.** Classify $f(x) = (x^2 - 16)/(x - 4)$ at $x = 4$.
   A. Hole (removable). Simplifies to $x + 4$, value 8.
4. **Q.** Find the value at the hole of $(x^2 - 25)/(x - 5)$.
   A. Factor → $x + 5$; at $x = 5$, value = 10.
5. **Q.** Find $\lim_{x \to 2^+} 1/(x - 2)$.
   A. $+\infty$.
6. **Q.** Find $\lim_{x \to 2^-} 1/(x - 2)$.
   A. $-\infty$.
7. **Q.** A function has left = 3 and right = 7 at $x = 0$. Does limit exist?
   A. No — jump discontinuity.
8. **Q.** Is $f(x) = \sqrt{x}$ continuous at $x = 0$?
   A. Yes from the right (domain $x \geq 0$).
9. **Q.** Find limit of $f(x) = (x - 1)/(x - 1)$ as $x \to 1$.
   A. 1 (function simplifies to 1 everywhere except $x = 1$ where there is a hole).
10. **Q.** Identify type: a sensor jumps from 5 V to 15 V at $t = 0$.
    A. Jump discontinuity.
11. **Q.** Identify type: a smooth temperature curve rising through 20°C.
    A. Smooth/continuous.
12. **Q.** Identify type: division by zero producing infinite voltage.
    A. Vertical asymptote.
13. **Q.** A table gives $f(2.9) = 5.81$, $f(2.99) = 5.98$, $f(3.01) = 6.02$, $f(3.1) = 6.21$. Estimate the limit.
    A. 6.
14. **Q.** Why does $(x - 2)/(x - 2)$ have a hole at $x = 2$ even though it simplifies to 1?
    A. Original function is undefined at $x = 2$ (0/0). The simplified form 1 fills the hole.
15. **Q.** What is the limit of $1/x$ as x approaches infinity?
    A. 0.

Quiz Answers

1. **Answer: (D) Removable discontinuity (hole)**
   Reason: Hole = removable discontinuity.
2. **Answer: (A) Jump discontinuity**

Reason: Jump = left ≠ right.

3. **Answer: (B) Vertical asymptote**

   Reason: Denominator → 0, numerator ≠ 0 → asymptote.

4. **Answer: (D) Hole**

   Reason: Both numerator and denominator are zero → removable.

5. **Answer: (D) 4**

   Reason: Factor and cancel: $x + 2 \to 4$.

6. **Answer: (A) Everywhere**

   Reason: Polynomials are continuous on all real numbers.

7. **Answer: (D) 4**

   Reason: Both sides agree.

8. **Answer: (C) $+\infty$**

   Reason: From right, denominator is small positive → 1/positive ≈ $+\infty$.

ANSWER KEY

Week 35 — Limits of Algebraic Functions

Practice Exercises

1. **Q.** $\lim_{x \to 1} (x^2 + 2x) = ?$
   A. 1 + 2 = 3.
2. **Q.** $\lim_{x \to 0} (3x^2 + 5x - 1) = ?$
   A. −1.
3. **Q.** $\lim_{x \to 3} (x^2 - 9)/(x - 3) = ?$
   A. Factor: x + 3 → 6.
4. **Q.** $\lim_{x \to 4} (x^2 - 16)/(x - 4) = ?$
   A. Factor: x + 4 → 8.
5. **Q.** $\lim_{x \to 2} (x^3 - 8)/(x - 2) = ?$
   A. Diff cubes: $x^2 + 2x + 4 \to 12$.
6. **Q.** $\lim_{x \to -1} (x^2 + 2x + 1)/(x + 1) = ?$
   A. Factor: $(x + 1)^2 / (x + 1) = x + 1 \to 0$.
7. **Q.** $\lim_{x \to 5} (x^2 - 5x)/(x - 5) = ?$
   A. Factor: x(x − 5)/(x − 5) = x → 5.
8. **Q.** $\lim_{x \to 4} (\sqrt{x} - 2)/(x - 4) = ?$
   A. Rationalise: $1/(\sqrt{x} + 2) \to 1/4$.
9. **Q.** $\lim_{x \to 0} (\sqrt{x+1} - 1)/x = ?$
   A. Rationalise: $1/(\sqrt{x+1} + 1) \to 1/2$.
10. **Q.** Why is direct substitution always tried first?
    A. For continuous functions, the limit equals the function value. Substitution gives the answer immediately when it works.
11. **Q.** $\lim_{x \to 2} (x^2 - 3x + 2)/(x - 2) = ?$
    A. Factor: (x−2)(x−1)/(x−2) = x − 1 → 1.
12. **Q.** $\lim_{x \to 3} (x^2 - 4x + 3)/(x - 3) = ?$
    A. Factor: (x−3)(x−1)/(x−3) = x − 1 → 2.
13. **Q.** If both numerator and denominator → 0 at x = a, what does that tell you?
    A. Both contain (x − a) as a factor. Factoring will let you cancel and find the limit.
14. **Q.** $\lim_{x \to 5} (x - 5)/(x^2 - 25) = ?$
    A. 1/(x + 5) → 1/10.
15. **Q.** $\lim_{x \to -2} (x^2 - 4)/(x + 2) = ?$
    A. Factor: (x − 2)(x + 2)/(x + 2) = x − 2 → −4.

Quiz Answers

1. **Answer: (D) 3**
   Reason: Direct sub: 1 + 2 = 3.
2. **Answer: (A) 6**

Reason: Factor: $x + 3 \to 6$.

3. **Answer: (A) 8**

   Reason: Factor: $x + 4 \to 8$.

4. **Answer: (A) 3**

   Reason: Factor: $x + 1 \to 3$.

5. **Answer: (D) Indeterminate — simplify and re-evaluate**

   Reason: 0/0 means "do more algebra."

6. **Answer: (A) Direct substitution**

   Reason: No 0/0 form — just plug in.

7. **Answer: (C) Rationalise (conjugate)**

   Reason: √-difference + 0/0 → use conjugate.

8. **Answer: (B) 10**

   Reason: Factor: $x + 5 \to 10$.

ANSWER KEY

Week 36 — Applications & Final Review

Practice Exercises

1. **Q.** Distance between (1, 2) and (5, 6)?
   A. $\sqrt{(16 + 16)} = \sqrt{32} \approx 5.66$.
2. **Q.** Midpoint of (3, 7) and (9, 1)?
   A. (6, 4).
3. **Q.** Slope between (2, 3) and (5, 9)?
   A. $(9 - 3)/(5 - 2) = 2$.
4. **Q.** Solve $x^2 - 5x + 6 = 0$.
   A. $(x - 2)(x - 3) = 0 \rightarrow x = 2$ or $x = 3$.
5. **Q.** Evaluate $\sin(\pi/3)$.
   A. $\sqrt{3}/2$.
6. **Q.** Evaluate $\cos(\pi/4)$.
   A. $\sqrt{2}/2$.
7. **Q.** $\log_2(16) = ?$
   A. 4.
8. **Q.** $2^3 \times 2^4 = ?$
   A. $2^7 = 128$.
9. **Q.** 10th term of $a_1 = 2$, $d = 4$?
   A. $2 + 9 \cdot 4 = 38$.
10. **Q.** Sum of first 10 terms of $a_1 = 1$, $d = 2$?
    A. $a_{10} = 19$. $S = 10/2 \cdot 20 = 100$.
11. **Q.** Identify $x^2 + y^2 = 25$.
    A. Circle of radius 5 centred at origin.
12. **Q.** Identify $y = x^2 - 4x + 3$.
    A. Parabola opening up. Vertex at (2, −1).
13. **Q.** $\lim_{x \to 3} (x + 2) = ?$
    A. 5.
14. **Q.** $\lim_{x \to 4} (x^2 - 16)/(x - 4) = ?$
    A. 8 (factor and cancel).
15. **Q.** A bag has 4 red, 3 blue, 3 green. Find P(not red).
    A. $6/10 = 3/5$.
16. **Q.** How many ways to arrange the letters in MATH?
    A. $4! = 24$.
17. **Q.** Find sum 1+2+3+…+100.
    A. 5050.
18. **Q.** A salary starts at $40,000 with annual raises of $2,000. Salary in year 10?
    A. 40000 + 9·2000 = $58,000.
19. **Q.** Total earnings over 10 years (sequence problem).

A. $a_1 = 40000$, $a_{10} = 58000$. S = 10/2 · 98000 = $490,000.

20. **Q.** A right triangle has angles 30°, 60°, 90° and hypotenuse 10. Find the shorter leg.
    A. 10 · sin(30°) = 5.

Quiz Answers

1. **Answer: (C) 5**
   Reason: √(9 + 16) = √25 = 5.
2. **Answer: (B) 32**
   Reason: 5 + 9·3 = 32.
3. **Answer: (D) 13**
   Reason: √(25 + 144) = √169 = 13.
4. **Answer: (B) 4/6**
   Reason: Outcomes 3, 4, 5, 6 = 4 favourable of 6.
5. **Answer: (D) 10**
   Reason: Factor: x + 5 → 10.
6. **Answer: (C) Exponential**
   Reason: Variable in the exponent → exponential.
7. **Answer: (B) Ellipse**
   Reason: Different denominators with + → ellipse.
8. **Answer: (D) 5050**
   Reason: S = 100·101/2 = 5050.